Hoofdstuk 4: Laplacetransformatie

laplacetransfo.

① c) $f(t) = 3/3^t = 3 \cdot 3^{-t} = 3 \cdot e^{\ln 3^{-t}} = 3\, e^{-t \cdot \ln 3)}$

$$\mathcal{L}[f(t)] = 3 \cdot \frac{1}{s+\ln 3} = \frac{3}{s+\ln 3}$$

$$\left[\begin{array}{l} \mathcal{L}[e^{\alpha t}] = \dfrac{1}{s-\alpha} \\ \alpha = -\ln(3) \end{array}\right.$$

inverse laplacetransfo.

② e) $F(s) = -\dfrac{2s-1}{s^2+4s+5} = \dfrac{1-2s}{(s+2)^2+1}$

$\quad \hookrightarrow (s+2)^2+1 \qquad \Delta < 0$

$$\left[\begin{array}{l} \mathcal{L}[e^{\alpha t}\cos at] = \dfrac{s-\alpha}{(s-\alpha)^2+a^2} \\ \mathcal{L}[e^{\alpha t}\sin at] = \dfrac{a}{(s-\alpha)^2+a^2} \end{array}\right.$$

$$= \frac{-2(s+2-2)+1}{(s+2)^2+1} = -2\,\frac{(s+2)}{(s+2)^2+1} + 5 \cdot \frac{1}{(s+2)^2+1}$$

$$\Rightarrow f(t) = -2 \cdot e^{-2t}\cos t + 5e^{-2t}\sin t = e^{-2t}(-2\cos t + 5\sin t)$$

opmerking

↳ opl noemer met neg. Δ schrijven als 2 complexe getallen geschreven met cos a sin

⑥ e) $f(t) = (t-3)\,u_2(t) + (t-2)\,u_3(t)$

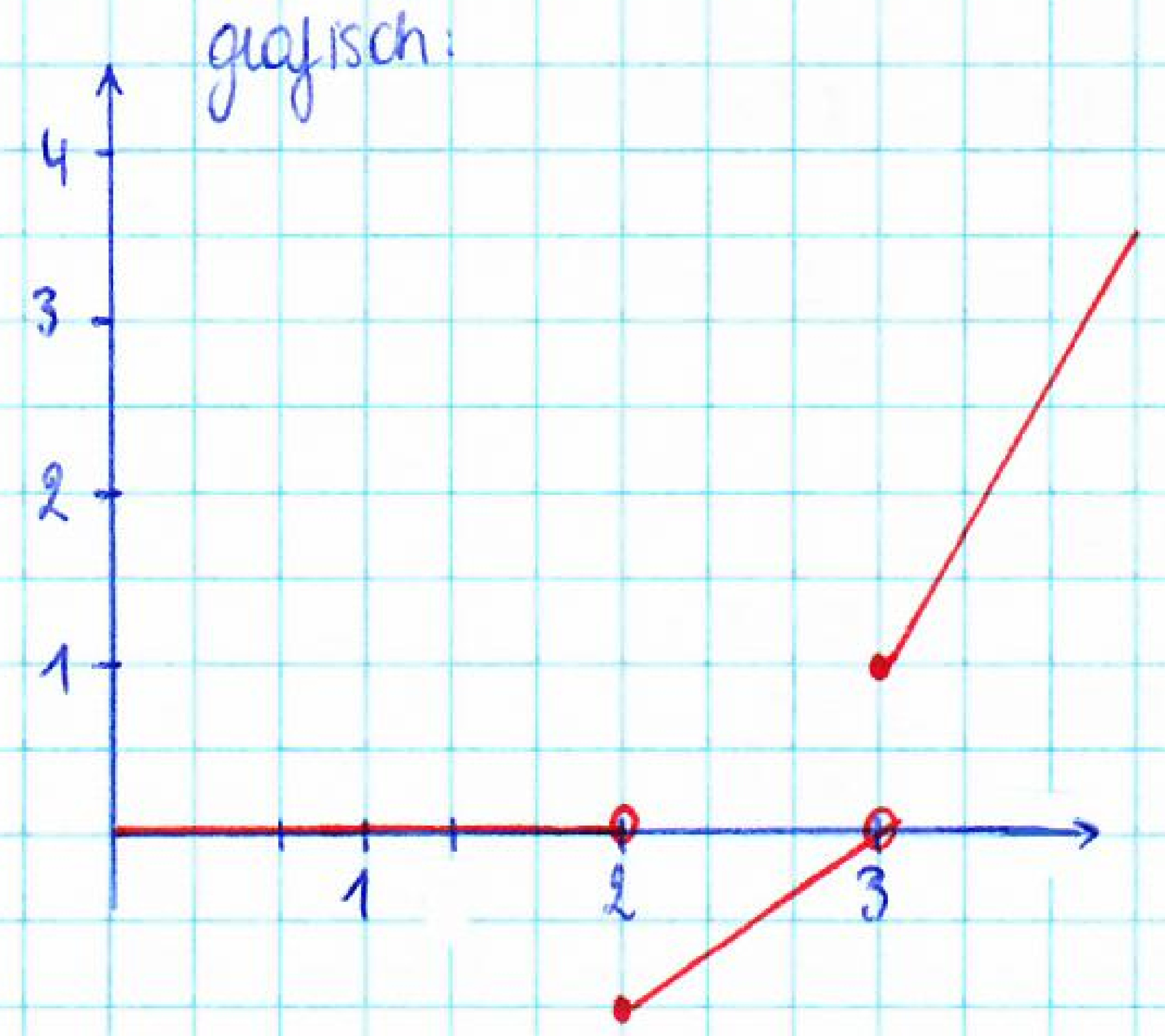
$$\left[\; u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \qquad t \geq 0 \right.$$

$$f(t) = \begin{cases} 0 & 0 \leq t < 2 \\ t-3 & 2 \leq t < 3 \\ 2t-5 & 3 \leq t \end{cases}$$

$\Rightarrow \mathcal{L}[(t-3)u_2(t)] + \mathcal{L}[(t-2)u_3(t)]$

$\left[\mathcal{L}[u_c(t)f(t-c)]\right] = e^{-cs}F(s)$

grafisch:



1e manier (substitutie)

- $f_1(t-2) = t-3$

  $f_1(t) = (t+2)-3 = t-1$

  $\mathcal{L}[f_1(t)] = F_1(s) = \dfrac{1}{s^2} - \dfrac{1}{s}$

- $f_2(t-3) = t-2$

  $f_2(t) = (t+3)-2 = t+1$

  $F_2(s) = \dfrac{1}{s^2} + \dfrac{1}{s}$

$$\Rightarrow \mathcal{L}[f(t)] = e^{-2s} \cdot \left(\frac{1}{s^2} - \frac{1}{s}\right) + e^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$= \frac{e^{-2s} + e^{-3s}}{s^2} + \frac{e^{-3s} - e^{-2s}}{s}$$

2e manier {

$$\mathcal{L}[f(t)] = \mathcal{L}[(t-2)u_2(t) - u_2(t) + (t-3)u_3(t) + u_3(t)]$$

$$= \frac{1}{s^2}\cdot(e^{-2s} + e^{-3s}) - \underbrace{\mathcal{L}[1]}_{1/s}\cdot(e^{-2s} - e^{-3s}) \qquad \mathcal{L}[u_c(t)] = \mathcal{L}[1]$$

7 a) $F(s) = (s-2)^{-4}$ $\qquad \mathcal{L}[t^n e^{\alpha t}] = \frac{n!}{(s-\alpha)^{n+1}}$

$$\mathcal{L}^{-1}\left[\frac{1}{6}\frac{\overset{3!}{\cdot 6}}{(s-2)^4}\right] = \frac{1}{6}t^3 \cdot e^{2t}$$

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2e manier (verschuivingsregel)

$$\mathcal{L}[e^{\gamma t}f(t)] = F(s-\gamma)$$

$$F(s-2) = \frac{1}{(s-2)^4} \Rightarrow F(s) = \frac{1}{s^4} \qquad \left[\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}\right.$$

$\gamma = 2$ $\qquad f(t) = t^3$

$$\mathcal{L}^{-1}[F(s-2)] = e^{2t}\cdot\frac{1}{6}\mathcal{L}^{-1}\left[\frac{3!}{s^4}\right]$$

$$= \frac{1}{6}t^3 \cdot e^{2t}$$

9 b) $\begin{cases} R' = 2R - 2\theta + e^{-t} \\ \theta' = 6R - 5\theta + e^{-2t} \end{cases}$ stelsel DV $R(0) = 1$; $\theta(0) = -1$

gebruik voor beginwaarde problemen

3 fasen techniek

fase 1 $\begin{cases} \mathcal{L}[R'] = 2\mathcal{L}[R] - 2\mathcal{L}[\theta] + \mathcal{L}[e^{-t}] \\ \mathcal{L}[\theta'] = 6\mathcal{L}[R] - 5\mathcal{L}[\theta] + \mathcal{L}[e^{-2t}] \end{cases}$

$$\Leftrightarrow \begin{cases} sR(s) - R(0) = 2R(s) - 2\Theta(s) + \frac{1}{s+1} \\ s\Theta(s) - \theta(0) = 6R(s) - 5\Theta(s) + \frac{1}{s+2} \end{cases}$$

fase 2 beginwaarden invullen & oplossen naar $R(s)$ en $\Theta(s)$

$$\begin{cases} (s-2)R(s) + 2\Theta(s) = 1 + \frac{1}{s+1} \\ -6R(s) + (s+5)\Theta = -1 + \frac{1}{s+2} \end{cases} \longrightarrow \text{stelsel vn Cramer!} \quad \begin{array}{c|c} R & \Theta \\ (s+5) & 6 \\ -2 & (s-2) \end{array}$$

Vgl 1: $(s-2)(s+5)R(s) + 12R(s) = s+5 + \frac{s+5}{s+1} + 2 - 2\frac{1}{s+2}$

~~[illegible]~~ =

Vgl 1: $(s+1)(s+2)R(s) = s+7 + \frac{s+5}{s+1} - \frac{2}{s+2}$

$\Leftrightarrow R(s) = \frac{s+7}{(s+1)(s+2)} + \frac{s+5}{(s+1)^2(s+2)} - \frac{2}{(s+1)(s+2)^2}$

partieelbreuken

$\Leftrightarrow R(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2}$

$\rightarrow (s+7)(s+1)(s+2) + (s+5)(s+2) - 2(s+1)$

$= A(s+1)(s+2)^2 + B(s+2)^2 + C(s+1)^2(s+2) + D(s+1)^2$

$(s+7)(s+1)(s+2)$
$= (s+7)(s^2+4s+4)$
$= s^3 + 4s^2 + 4s + 7s^2 + 28s + 28$
$= s^3 + 11s^2 + 32s + 28$

$(s+5)(s+2) = s^2 + 7s + 10$

$-2s - 2$

$\Rightarrow s^3 + 12s^2 + 37s + 36$

$s = -1 \rightarrow 4.1 = B.(1^2) \qquad B = 4$

$s = -2 \rightarrow -2.(-1) = D(-1)^2 \qquad D = 2$

$s = 0 \rightarrow 14 + 10 - 2 = 4A + 4B + 2C + D$

$s^3 \rightarrow 1 = A + C$

$\begin{cases} 4 = 4A + 2C & \Leftrightarrow 4 = 4(1-C) + 2C \\ 1 = A + C & \Leftrightarrow 0 = -2C \end{cases}$

$\begin{cases} A = 1 \\ B = 4 \\ C = 0 \\ D = 2 \end{cases}$

$R = \frac{1}{s+1} + \frac{4}{(s+1)^2} + \frac{2}{(s+2)^2}$

fase 3 (R)

$\mathcal{L}^{-1}[R(s)] = e^{-t} + 4\, t.e^{-t} + 2.t.e^{-2t} = R(t)$

analoog voor $\Theta(s)$

Vgl 2: $12\,\Theta(s) + (s-2)(s+5)\Theta(s) = 6 + \frac{6}{s+1} - s + 2 + \frac{s-2}{s+2}$

$12 + s^2 + 3s - 10 = s^2 + 3s + 2$
$= (s+1)(s+2)$

$\Theta(s) = \frac{-s+8}{(s+1)(s+2)} + \frac{6}{(s+1)^2(s+2)} + \frac{(s-2)}{(s+1)(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)} + \frac{D}{(s+2)^2}$

$(8-s)(s+1)(s+2) + 6(s+2) + (s-2)(s+1) = A(s+1)(s+2)^2 + B(s+2)^2 + C(s+1)^2(s+2) + D(s+1)^2$

$s = -1 \qquad 6 = B$

$s = -2 \qquad +4 = D$

$s = 0 \qquad 16 + 12 - 2 = 4A + \overset{24}{4B} + 2C + \overset{4}{D}$

$-2 = 4A + 2C \qquad \cancel{-2} = 4A \cancel{-2} - 2A \Leftrightarrow 2A = 0$

$s^3 \qquad -1 = A + C$

$\begin{cases} A = 0 \\ C = -1 \\ B = 6 \\ D = +4 \end{cases}$

$$\Theta(s) = \frac{0}{s+1} + \frac{6}{(s+1)^2} - \frac{1}{s+2} + \frac{4}{(s+2)^2}$$

$$\mathcal{L}^{-1}[\Theta(s)] = \theta(s) = 6.\ te^{-t} - e^{-2t} + 4t.e^{-2t}$$

(10) e) $$v'' = v - \begin{cases} \sin t & t \in [0, \frac{\pi}{2}[\\ 0 & t \in [\frac{\pi}{2}, +\infty[\end{cases} \qquad v(0) = v'(0) = 0$$

$$v'' = v - \sin t + (\sin t)\, u_{\pi/2}(t)$$

$$\mathcal{L}[v''] = \mathcal{L}[v] - \mathcal{L}[\sin(t)] + \mathcal{L}[u_{\pi/2}(t) . \overset{\cos(t-\frac{\pi}{2})}{\sin t}] \text{ verschuivingsregel}$$

$$s^2 V(s) - s v(0) - v'(0) = V(s) - \frac{1}{s^2+1} + e^{-\pi/2 s} . \mathcal{L}[\cos(t - \cancel{\tfrac{\pi}{2}})]$$

$$s^2 V(s) - V(s) = \frac{-1}{s^2+1} + e^{-\pi/2 s} . \frac{s}{s^2+1}$$

<u>fase 2</u>

$$\cancel{(s^2-1)}V(s) = \underbrace{\frac{-1}{(s-1)(s^2+1)(s+1)}}_{F_1(s)} + \underbrace{\frac{s}{(s^2+1)(s-1)(s+1)}}_{F_2(s)} . e^{-\pi/2 s} \qquad (s+D)/s^2+1$$

$F_1(s)$: $-1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + Cs(s-1)(s+1) + D(s-1)(s+1)$

$s = -1$	$-1 = -4B$	$B = 1/4$
		$A = -1/4$
$s = 1$	$-1 = 4A$	$C = 0$
		$D = 1/2$
s^3	$0 = A + B + C$	
$0 = s$	$-1 = -1/2 - D$	

$$\mathcal{L}^{-1}[F_1(s)] = -\frac{1}{4}\mathcal{L}^{-1}[\frac{1}{s-1}] + \frac{1}{4}\mathcal{L}^{-1}[\frac{1}{s+1}] + \frac{1}{2}\mathcal{L}^{-1}[\frac{1}{s^2+1}]$$

$$f_1(t) = -\frac{1}{4}e^t + \frac{1}{4}e^{-t} + \frac{1}{2}\sin t$$

$$F_2(s) = \frac{s}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$s = A(s+1)(s^2+1) + B(s-1)(s^2+1) + Cs(s^2-1) + D(s^2-1)$$

$s = -1$ $\quad -1 =$