Hoofdstuk 2: Complexe getallen

(1)

a) $(2+3i)(16-3i) = 32 - 6i + 48i - 9i^2 = 41 + 42i$

b) $(2+\sqrt{5}i)(3-2\sqrt{5}i) = 6 - 4\sqrt{5}i + 3\sqrt{5}i - 10i^2 = 16 - \sqrt{5}i$

c) $(2-i)^2 i = (4 - 4i - 1)i = 3i + 4 = 4 + 3i$

d) $(\sqrt{13} - 2i)^{-1} = \dfrac{1}{\sqrt{13}-2i} \cdot \dfrac{\sqrt{13}+2i}{\sqrt{13}+2i} = \dfrac{\sqrt{13}+2i}{13+4} = \dfrac{1}{17}(\sqrt{13}+2i)$

e) $\dfrac{3+i}{i} \cdot \dfrac{i}{i} = -1(3i-1) = 1-3i$

f) $\dfrac{3\sqrt{2}-2i}{4-\sqrt{2}i} \cdot \dfrac{4+\sqrt{2}i}{4+\sqrt{2}i} = \dfrac{1}{16+2} \cdot (3\sqrt{2}-2i)(4+\sqrt{2}i) = \dfrac{1}{18}(12\sqrt{2} + 6i - 8i + 2\sqrt{2})$

$= \dfrac{1}{18}(14\sqrt{2} - 2i) = \dfrac{7\sqrt{2}-i}{9}$

g) $\sqrt{7+6\sqrt{2}i}$ vierkantswortels

$(x+iy)^2 = 7 + 6\sqrt{2}i$

$x^2 - y^2 + 2xyi = 7 + 6\sqrt{2}i$

$\begin{cases} x^2 - y^2 = 7 \\ 2xy = 6\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} \dots \\ xy = 3\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} x^2 - \dfrac{9.2}{x^2} = 7 \\ y = \dfrac{3}{x}\sqrt{2} \end{cases}$

$\Leftrightarrow x^4 - 7x^2 - 18 = 0$

$\Delta = 49 + 4.18 = 121$

$x_1 = \dfrac{7+11}{2} = 9 \wedge y = \dfrac{\sqrt{2}}{3}$ $\qquad 9 + \dfrac{\sqrt{2}}{3}i$

OK $x_2 = -2 \qquad \wedge y = -\dfrac{3\sqrt{2}}{2}$ $\qquad -2 - \dfrac{3\sqrt{2}}{2}i$

$\hookrightarrow x_1^2 = 9 \Leftrightarrow x = \pm 3 \qquad y = \pm\sqrt{2}$

$x_2^2 \cancel{=} -2 \qquad 3+\sqrt{2}i \vee -3-\sqrt{2}i$

h) $-4+3i = (x+yi)^2$

$$\begin{cases} x^2 - y^2 = -4 \\ 2xy = 3 \end{cases} \Leftrightarrow \begin{cases} x^2 - \dfrac{9}{4x^2} = -4 \\ y = \dfrac{3}{2x} \end{cases} \Leftrightarrow \begin{cases} 4x^4 + 16x^2 - 9 = 0 \\ y = 3/2x \end{cases}$$

$\Delta = 256 + 4 \cdot 36 = 20^2$

$x_1^2 = \dfrac{-16+20}{8} = \dfrac{1}{2}$ $\quad x_1 = \pm \dfrac{\sqrt{2}}{2}$ $\quad y = \pm \dfrac{3\sqrt{2}}{2}$

$x_2^2 \cancel{=} \dfrac{-36}{8}$ $\quad \dfrac{\sqrt{2}}{2}(1 \pm 3i)$

i) $4-3i = (x+yi)^2$

$$\begin{cases} x^2 - y^2 = 4 \\ 2xy = -3 \end{cases} \Leftrightarrow \begin{cases} x^2 - 9/4x^2 = 4 \\ y = -3/2x \end{cases} \Leftrightarrow \begin{cases} 4x^4 - 16x^2 - 9 = 0 \\ \dots \end{cases}$$

$\Delta = 20^2$

$x_1^2 = \dfrac{16+20}{8} = \dfrac{9}{2}$ $\quad x_1 = \pm \dfrac{3\sqrt{2}}{2}$ $\quad y = \mp \dfrac{\sqrt{2}}{2}$

$x_2^2 \cancel{=} \dfrac{16-20}{8}$ $\quad \dfrac{\sqrt{2}}{2}(3-i) \vee \dfrac{\sqrt{2}}{2}(-3+i)$

eigenschappen toegevoegde complexe getallen

! $\dfrac{4+3i}{2} = \left(\overline{\dfrac{4-3i}{2}}\right) = \dfrac{\bar{z}^2}{2} = \left(\dfrac{\bar{z}}{\sqrt{2}}\right)^2$

$\Rightarrow \dfrac{\bar{z}}{\sqrt{2}} = \dfrac{3+i}{2} \vee -\dfrac{3-i}{2}$

EXAMEN ② polaire voorstelling

1) $\dfrac{(1+i)(\sqrt{3}+i)^3}{(1-i\sqrt{3})^3} \cdot \dfrac{i^3}{i^3}$

shortcut moet je zien; anders via de lange weg

polaire voorstelling & Moivre

$= \dfrac{(1+i)\cancel{(\sqrt{3}+i)^3}\, i^3}{\cancel{(i-i^2\sqrt{3})^3}} = -i\,(1+i) = 1-i$

❗ ③ oplossing in de vorm $a+ib$

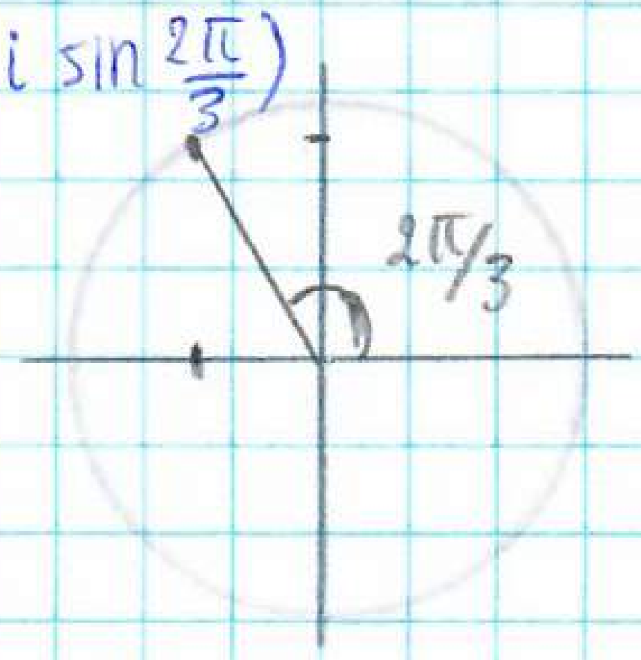
a) $z^4 + 8 - 8\sqrt{3}i = 0$

eerst afvragen: hoeveel oplossingen zullen er zijn voor deze vgl?

z^4 dus 4 oplossingen: vierdemacht

$z^4 = -8 + 8\sqrt{3}i$

$z^4 = 8\left(\dfrac{-1}{2}+\dfrac{\sqrt{3}i}{2}\right) = 16\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right) = 16\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$



of $R = \sqrt{(-1)^2+(\sqrt{3})^2} = \boxed{2} = |z|$

$\cos\theta = -1/2 = a/R$

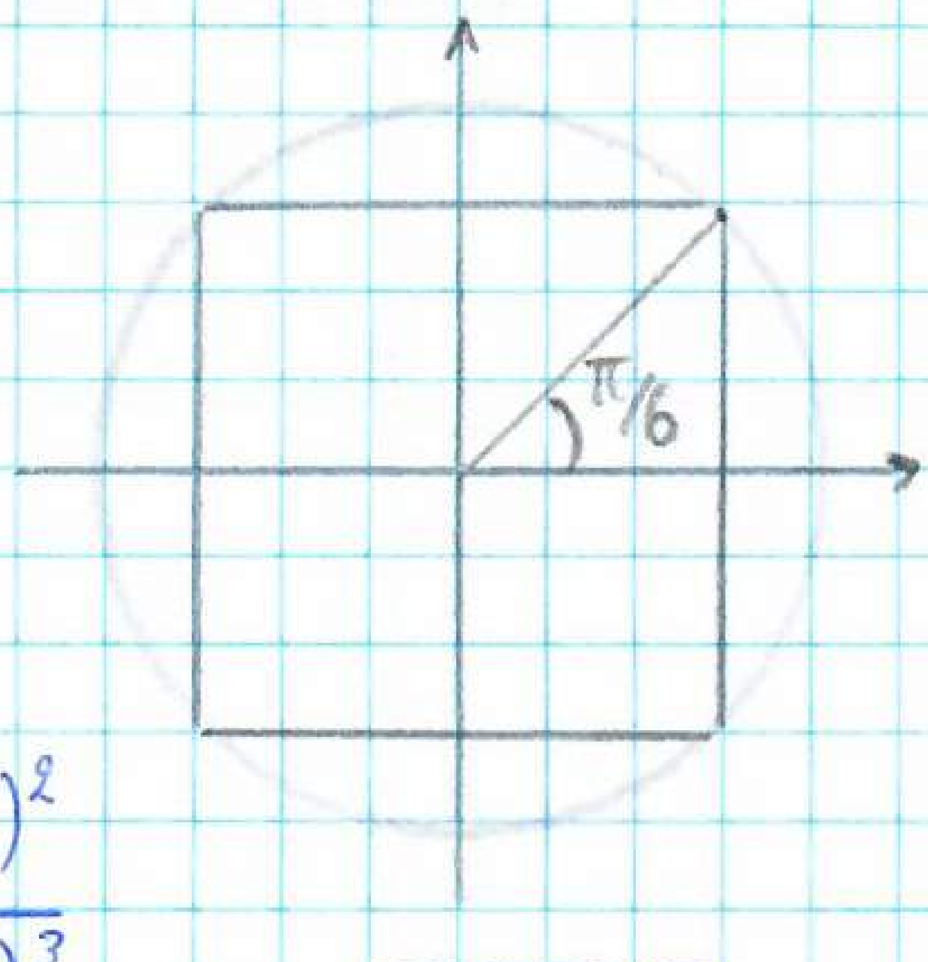
$\tan\theta = -\sqrt{3}$

$\sin\theta = \sqrt{3}/2 = b/R$

$\boxed{\theta = 2\pi/3}$ $\quad z^4 = 16e^{i\,2\pi/3}$

$k=0 \quad z = 2e^{i\pi/6} = \dfrac{2\sqrt{3}}{2} + i\dfrac{2}{2} = \sqrt{3}+i$

$z = \sqrt[4]{16}\, e^{i\left(\frac{\frac{2\pi}{3}+k2\pi}{4}\right)} \quad k = 0,1,2,3$



② a) $\dfrac{(-2+2\sqrt{3}i)^2}{(-1-i)^3} = \dfrac{4(-1+\sqrt{3}i)^2}{(-1)(1+i)^3} = -4\dfrac{\left(2e^{\frac{2\pi i}{3}}\right)^2}{\left(\sqrt{2}\cdot e^{\frac{\pi i}{4}}\right)^3}$

$\boxed{e^{i\pi} = -1}$

$= \dfrac{-4\sqrt{2}\, e^{\frac{4\pi i}{3}}}{e^{\frac{3\pi}{4}i}} = -4\sqrt{2}\, e^{\frac{7\pi i}{12}} = 4\sqrt{2}\cdot e^{i\pi}\cdot e^{7\pi i/12}$

$= 4\sqrt{2}\, e^{\frac{13\pi}{12}i} = 4\sqrt{2}\, e^{\frac{-5\pi}{12}i}$

③ b) $z^6 - 1 = 0$ → 6 nulpunten met 0, 2, 4, 6 reële nulpunten

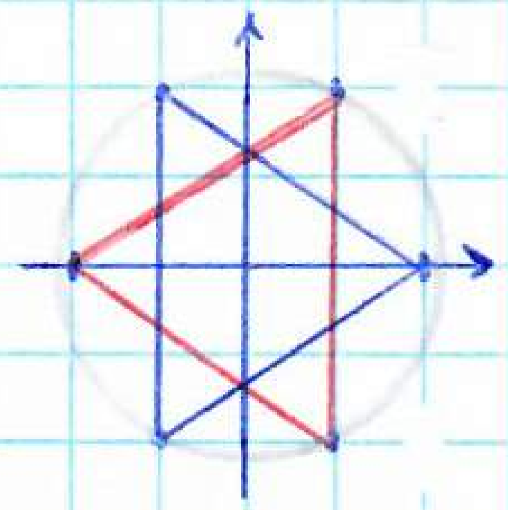
$z^6 = 1$ → $-1, 1$ = nulpunten
andere zoeken!

$1 = e^{i\pi \cdot 0}$

$\Rightarrow z = e^{k\pi i/3}$ $k = 0, \dots 5$

z^6 → $(-z^3)^2$, $(z^3)^2$

alle nulpunten van z^3 + gespiegelde punten



$z_0 = 1$

$z_1 = e^{\pi i/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_2 = \dots = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_3 = \dots = -1$

$z_4 = \dots = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$z_5 = \dots = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

zie schema op UFORA! ! g) $(z-1)^4 - \sqrt{2}(z-1)^2 + 2 = 0$

EXAMEN

$x^4 - \sqrt{2}x^2 + 2 = 0$ $(x = z-1)$

$y^2 - \sqrt{2}y + 2 = 0$ $(y = x^2)$

$\Delta = 2 - 4 \cdot 2 = -6$

$y_1 = \frac{\sqrt{2} + i\sqrt{6}}{2} = \sqrt{2}e^{i\pi/3}$

$y_2 = \frac{\sqrt{2} - i\sqrt{6}}{2} = \sqrt{2}e^{-i\pi/3} = \bar{y}_1$

y_1: $x_1, x_2 = -x_1$ y_2: $x_3 = \bar{x}_1, x_4 = -\bar{x}_1$

$z_1 = x_1 + 1$ $z_3 = \bar{x}_1 + 1$

$z_2 = -x_1 + 1$ $z_4 = -\bar{x}_1 + 1$

$z_1 = \left(1 + \frac{\sqrt[4]{2}\sqrt{3}}{2}\right) + \frac{\sqrt[4]{2}}{2}i = x_1 + 1$

$z_2 = (1 - \dots) - \dots = -x_1 + 1$

$z_3 = (1 + \dots) - \dots = \bar{x}_1 + 1$

$z_4 = (1 - \dots) + \dots = -\bar{x}_1 + 1$

(4) $|z - 3i| = 3$ $\quad\quad |z| = 3$ = alle punten op cirkel met straal 3

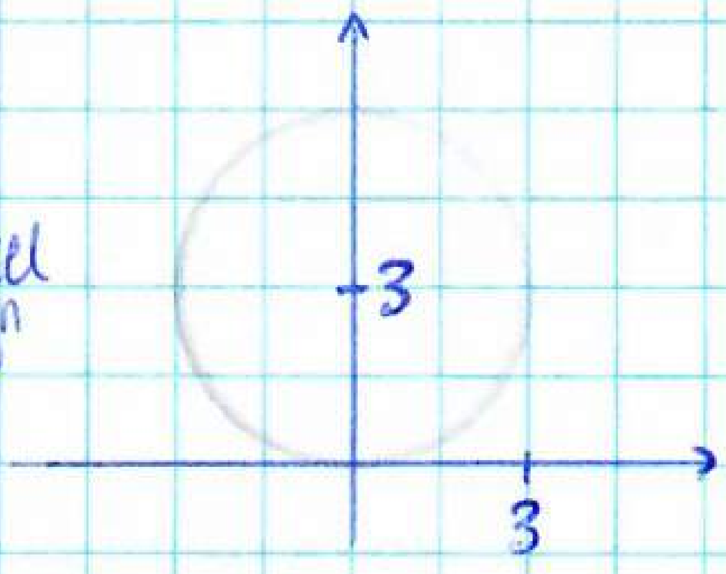
$\cancel{\sqrt{(z-3i)^2} = 3}$

$\cancel{+\sqrt{(z-3i)^2} = +3}$ $\quad\quad |z-3i| = |z-3|$? theorie

$z = x + yi$

$\Leftrightarrow \sqrt{x^2 + (y-3)^2} = 3$

$\Leftrightarrow x^2 + (y-3)^2 = 9 \rightarrow$ cirkel vglⁿ $\quad\quad$ ... verder uitwerken



3 nulpunten

(6) $z^3 - 31z^2 + 68z - 290 = 0$

$(z - z_1)(z - z_2)(z - z_3) = 0$ $\quad\quad z^3 - 31$

$$\left.\begin{array}{ll} z_1 = 1 + bi & \in \mathbb{C} \\ z_2 = 1 - bi & \in \mathbb{C} \\ z_3 = R & \in \mathbb{R} \end{array}\right| \; b, R \;?$$

$1 - (bi)^2 = 0$

$z = R$

$(z^2 - (z_1 + \bar{z}_1)z + z_1 . \bar{z}_1)(z - R) = 0$

$(z^2 - 2\underbrace{(\mathrm{Re}\, z_1)}_{=1} z + |z_1|)(z - R) = 0$

$(z^2 - 2z + (1 + b^2))(z - R) = 0$

$$\begin{array}{l} \text{0e graad} \\ \text{1e graad} \\ \text{2e graad} \end{array} \begin{cases} +R(1+b^2) = +290 \Leftrightarrow 1 + b^2 = 10 \Rightarrow b = \pm 3 \\ (1 + b^2) + 2R = 68 \\ +2 + R = +31 \Rightarrow R = 29 \end{cases}$$

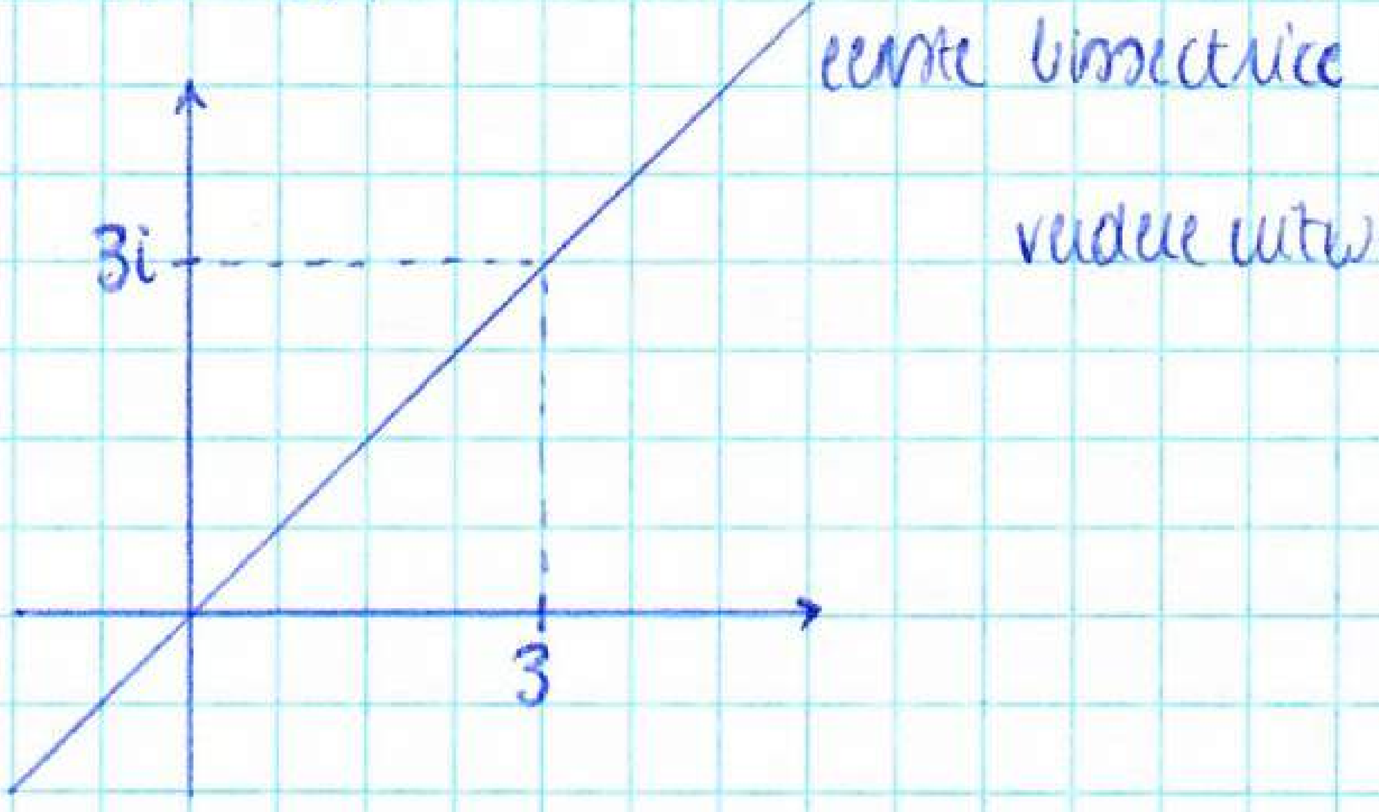
theorie & eigenschappen echt nodig voor oefeningen

$\Leftrightarrow (z - (1+3i))(z - (1-3i))(z - 29) = 0$

opl: $\begin{cases} z_1 = 1 + 3i \\ z_2 = 1 - 3i \\ z_3 = 29 \end{cases}$

EXAMEN VRAAG
voorbeeld

$|z-3| = |z-3i|$



verdere uitwerking zelf ...