Onopgeloste oefeningen

H1 23

H2 2.1 24

2.3 3

2.5 7

H3 3.3 4

3.4

3.8

3.9

H4 4.6

H5 5.6

H6 /

H7

Als jullie fouten vinden en/of de rest van de oefeningen kunnen oplossen; deel het dan in de groep!

Astublieft, graag gedaan en succes!

Hoofdstuk 1: Integraalrekening

(1.1) ① $\int \ln(3x)\,dx$

$f(x) = \ln(3x) \qquad f'(x) = \frac{1}{3x}\cdot 3 = \frac{1}{x}$

$g'(x) = 1 \qquad g(x) = x$

$\Rightarrow \ln(3x).x - \int \frac{1}{x}.x\,dx$

$= \ln(3x).x - x + C$

② $\int x^3.\ln x\,dx$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g'(x) = x^3 \qquad g(x) = \frac{x^4}{4}$

$\Rightarrow \frac{\ln x.x^4}{4} - \frac{1}{4}\int x^3 dx + C$

$= \frac{x^4}{4}\left[\ln x - \frac{1}{4}\right] + C$

③ $\int x^2.\cos(3x)\,dx$

$f(x) = x^2 \qquad f'(x) = 2x$

$g'(x) = \cos(3x) \qquad g(x) = \frac{\sin 3x}{3}$

$\Rightarrow \frac{1}{3}x^2.\sin 3x - \frac{2}{3}\int x.\sin 3x$

$f(x) = x \qquad f'(x) = 1$

$g'(x) = \sin 3x \qquad g(x) = -\frac{\cos 3x}{3}$

$\Rightarrow \frac{1}{3}x^2.\sin 3x - \frac{2}{3}\left(\frac{1}{3}x.(-\cos 3x) + \frac{1}{3}\int \cos 3x\right)$

$= \frac{1}{3}x^2.\sin 3x + \frac{2}{9}x.\cos 3x - \frac{2}{27}\sin 3x + C$

④ $\int \frac{dx}{x^2+4} = \int \frac{dx}{4\left(\frac{x^2}{4}+1\right)}$

$= \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1}$

$= \frac{1}{4} \cdot 2 \cdot \text{Bgtan}\,\frac{x}{2} + C$

$= \frac{1}{2} \cdot \text{Bgtan}\,\frac{x}{2} + C$

⑤ $\int x^8 \sqrt{x^3+1}\, dx$ stel $u = x^3+1$

$du = 3x^2 dx$

$= \int \frac{1}{3}(u-1)(u-1)\sqrt{u}\, du$

$= \frac{1}{3}\int (u^2-2u+1)\sqrt{u}\, du$

$= \frac{1}{3}\left(\int u^2\sqrt{u}\, du - \int 2u\sqrt{u}\, du + \int \sqrt{u}\, du\right)$

$= \frac{1}{3}\left(\int u^{5/2} du - 2\int u^{3/2} du + \int u^{1/2} du\right)$

$= \frac{1}{3} \cdot \frac{2}{7} \cdot u^{7/2} - 2 \cdot \frac{1}{3} \cdot \frac{2}{5} \cdot u^{5/2} + \frac{2}{3} \cdot \frac{1}{3} \cdot u^{3/2} + C$

$= \frac{2}{21}(x^3+1)^{7/2} - \frac{4}{15}(x^3+1)^{5/2} + \frac{2}{9}(x^3+1)^{3/2} + C$

⑥ $\int \sqrt{1-x^2}\, dx$

$f(x) = \sqrt{1-x^2}$ $\quad f'(x) = \frac{-x}{\sqrt{1-x^2}}$

$g'(x) = 1$ $\quad g(x) = x$

$\Rightarrow x \cdot \sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\, dx$

$= x\sqrt{1-x^2} + \int \frac{x^2-1+1}{\sqrt{1-x^2}}\, dx$

$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\, dx + \text{Bgsin}\, x + C$

$I + I = x \cdot \sqrt{1-x^2} + \text{Bgsin}\, x + C$

$I = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\text{Bgsin}\, x + C$

⑦ $\int \sqrt{a^2 - x^2}\, dx \quad (a > 0)$

$= a \int \sqrt{1 - \left(\frac{x}{a}\right)^2}\, dx$

→ zelfde methode als vorige oefening

⑧ $\int \frac{dx}{x\sqrt{1 - \ln^2 x}}$ $\qquad$ stel $u = \ln x$

$\qquad\qquad du = \frac{1}{x} dx$

$= \int \frac{du}{\sqrt{1 - u^2}}$

$= \text{Bgsin}\, u + C$

$= \text{Bgsin}(\ln x) + C$

⑨ $\int \frac{\ln\sqrt{x}}{x\sqrt{3 + 2\ln x}}\, dx$ $\qquad$ stel $u = 3 + 2\ln x$ $\quad$ → $\ln x = \frac{u-3}{2}$

$\qquad du = 2\frac{1}{x} dx \qquad \ln\sqrt{x} = \frac{1}{2}\ln x = \frac{u-3}{4}$

$= \frac{1}{2} \int \frac{\frac{u-3}{4}}{\sqrt{u}}\, du$

$= \frac{1}{8}\left(\int \frac{u}{\sqrt{u}}\, du - \int \frac{3}{\sqrt{u}}\, du\right)$

$= \frac{1}{8}\left(\int u^{1/2} du - 3\int u^{-1/2} du\right)$

$= \frac{1}{8} \cdot \frac{2}{3} u^{3/2} - \frac{1}{8} \cdot 3 \cdot 2 u^{1/2} + C$

$= \frac{1}{12}(3 + 2\ln x)^{3/2} - \frac{3}{4}(3 + 2\ln x)^{1/2} + C$

⑩ $\int \frac{e^{2x}}{e^x + 5}\, dx$ $\qquad$ stel $u = e^x + 5$

$\qquad du = e^x dx$

$= \int \frac{(u-5)}{u}\, du$

$= \int du - \int \frac{5}{u}\, du$

$= u - 5\ln u + C$

$= e^x + \cancel{5} - 5\ln(e^x + 5) + C$

(11) $\int e^x \cdot \cos(2-x)\,dx$

$$f(x) = \cos(2-x) \qquad f'(x) = \sin(2-x)$$
$$g'(x) = e^x \qquad g(x) = e^x$$

$\Rightarrow e^x \cdot \cos(2-x) - \int e^x \sin(2-x)$

$$f(x) = \sin(2-x) \qquad f'(x) = -\cos(2-x)$$
$$g'(x) = e^x \qquad g(x) = e^x$$

$\Rightarrow e^x \cos(2-x) - \left(e^x \sin(2-x) + \int e^x \cos(2-x)\right)$

$I + I = e^x \cos(2-x) - e^x \sin(2-x) + C$

$I = \frac{e^x}{2} \cos(2-x) - \frac{e^x}{2} \sin(2-x) + C$

(12) $\int e^{5x+1} \sin(1-x)\,dx$

$$f(x) = e^{5x+1} \qquad f'(x) = 5 \cdot e^{5x+1}$$
$$g'(x) = \sin(1-x) \qquad g(x) = \cos(1-x)$$

$\Rightarrow e^{5x+1} \cos(1-x) - 5\int e^{5x+1} \cos(1-x)$

$$f(x) = e^{5x+1} \qquad f'(x) = 5e^{5x+1}$$
$$g'(x) = \cos(1-x) \qquad g(x) = -\sin(1-x)$$

$\Rightarrow e^{5x+1} \cos(1-x) - 5\left(-e^{5x+1} \sin(1-x) + 5\int e^{5x+1} \sin(1-x)\right)$

$= e^{5x+1} \cos(1-x) + 5 \cdot e^{5x+1} \sin(1-x) - 25 I$

$26 I = \ldots$

$I = \frac{1}{26} e^{5x+1} \cos(1-x) + \frac{5}{26} \cdot e^{5x+1} \sin(1-x) + C$

(13) zelfde methode als vorige 2 oefeningen

(14) $$\int \frac{x - \mathrm{Bgtan}\,x}{1+x^2}\,dx$$

$$= \int \frac{x}{1+x^2}\,dx - \int \frac{\mathrm{Bgtan}\,x}{1+x^2}\,dx \qquad \text{stel } u = \mathrm{Bgtan}\,x$$

$$du = \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{t}\,dt - \int u\,du$$

$$\text{stel } t = 1+x^2$$

$$= \frac{1}{2}\ln t - \frac{1}{2}u^2 + C$$

$$dt = 2x\,dx$$

$$= \frac{1}{2}\ln(1+x^2) - \frac{1}{2}\mathrm{Bgtan}^2 x + C$$

(15) $$\int \sin(\ln x)\,dx$$

$$f(x) = \sin(\ln x) \qquad f'(x) = \cos(\ln x)\,\frac{1}{x}$$
$$g'(x) = 1 \qquad g(x) = x$$

$$\Rightarrow \sin(\ln x)\,x - \int \cos(\ln x)\,dx$$

$$f(x) = \cos(\ln x) \qquad f'(x) = -\sin(\ln x)\,\frac{1}{x}$$
$$g'(x) = 1 \qquad g(x) = x$$

$$\Rightarrow \sin(\ln x).x - (\cos(\ln x)\,x + \int \sin(\ln x)\,dx$$

$$2I = \sin(\ln x).x - \cos(\ln x).x + C$$

$$I = \frac{x}{2}.\sin(\ln x) - \frac{x}{2}.\cos(\ln x) + C$$

(16) $$\int \frac{\tan(\ln(1-x))}{1-x}\,dx \qquad \text{stel } u = \ln(1-x)$$

$$du = -\frac{1}{1-x}\,dx$$

$$= -\int \tan u\,du$$

$$= -\int \frac{\sin u}{\cos u}\,du \qquad \text{stel } t = \cos u$$

$$dt = -\sin u\,du$$

$$= \int \frac{dt}{t}$$

$$= \ln t + C = \ln(\cos u) + C$$

$$= \ln(\cos(\ln(1-x))) + C$$

⑰ $\int \frac{1+\sin(2x)}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx + \int \frac{\sin(2x)}{\sin^2 x}\,dx$

$= -\cot x + \int \frac{2\sin x\cos x}{\sin^2 x}\,dx$

$= -\cot x + \int \frac{du}{u}$ stel $u = \sin^2 x$, $du = 2\sin x\cos x\,dx$

$= -\cot x + \ln|u| + C$

$= -\cot x + \ln|\sin^2 x| + C$

$= -\cot x + 2\ln|\sin x| + C$

⑱ $\int \frac{\ln x}{\sqrt{x}}\,dx$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g'(x) = x^{-1/2} \qquad g(x) = 2x^{1/2}$

$= 2\sqrt{x}\,\ln x - 2\int \frac{1}{x}\,x^{1/2}\,dx$

$= 2\sqrt{x}\,\ln x - 2\int x^{-1/2}\,dx$

$= 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$

⑲ $\int (x-2)\,a^x\,dx$

$f(x) = (x-2) \qquad f'(x) = 1$

$g'(x) = a^x \qquad g(x) = \frac{a^x}{\ln a}$

$= \frac{(x-2)\,a^x}{\ln a} - \frac{1}{\ln a}\int a^x\,dx$

$= \frac{(x-2)\,a^x}{\ln a} - \frac{1}{\ln a}\,\frac{a^x}{\ln a} + C$

$= \frac{a^x}{\ln a}\left(x-2-\frac{1}{\ln a}\right) + C$

(20) $\displaystyle\int \frac{dx}{x^2+a^2} = \int \frac{1}{a^2} \cdot \frac{1}{\left(\frac{x}{a}\right)^2+1}\, dx$ stel $u = \frac{x}{a}$, $du = \frac{1}{a}\,dx$

$\displaystyle = \frac{1}{a}\int \frac{du}{u^2+1}$

$\displaystyle = \frac{1}{a} \cdot \mathrm{Bgtan}\, u + C$

$\displaystyle = \frac{1}{a} \cdot \mathrm{Bgtan}\, \frac{x}{a} + C$

(21) $\displaystyle\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{x^2+2x+1+4}$

$\displaystyle = \int \frac{dx}{(x+1)^2+4}$

$\displaystyle = \frac{1}{4}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1}$ stel $u = \frac{x+1}{2}$, $du = \frac{1}{2}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{du}{u^2+1}$

$\displaystyle = \frac{1}{2} \cdot \mathrm{Bgtan}\, u + C$

$\displaystyle = \frac{1}{2} \cdot \mathrm{Bgtan}\left(\frac{x+1}{2}\right) + C$

(22) $\displaystyle\int \frac{dx}{4x^2-4x+5} = \int \frac{dx}{(2x-1)^2+4}$

$\displaystyle = \frac{1}{4}\int \frac{dx}{\left(\frac{2x-1}{2}\right)+1}$ stel $u = \frac{2x-1}{2}$, $du = dx$

$\displaystyle = \frac{1}{4}\int \frac{dx}{u^2+1}$

$\displaystyle = \frac{1}{4}\,\mathrm{Bgtan}\, u + C$

$\displaystyle = \frac{1}{4}\,\mathrm{Bgtan}\left(\frac{2x-1}{2}\right) + C$

(23)

(24) $\int \frac{dx}{x^2-4x+4} = \int \frac{dx}{(x-2)^2}$ stel $u = x-2$

$du = dx$

$= \int \frac{du}{u^2}$

$= \int u^{-2}\,du$

$= -u^{-1} + C$

$= -\frac{1}{x-2} + C$

Hoofdstuk 2: Bepaalde en oneigenlijke integralen

(2.1) ① $\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$
$= -(\cos\pi - \cos 0)$
$= -(-1-1)$
$= 2$

② $\int_2^4 t^2 dt = \left[\frac{t^3}{3}\right]_2^4$
$= \frac{64}{3} - \frac{8}{3}$
$= \frac{56}{3}$

③ $\int_e^1 \frac{dx}{x} = [\ln|x|]_e^1$
$= \ln 1 - \ln e$
$= 0 - 1$
$= -1$

④ $\int_1^5 \sqrt{x}\, dx = \left[\frac{2}{3} x^{3/2}\right]_1^5$
$= \frac{2}{3}(5^{3/2} - 1)$
$= \frac{2}{3}(5\sqrt{5} - 1)$

⑤ $\int_0^1 x(3x^2+4)^3 dx$ Stel $u = 3x^2+4$
$du = 6x\,dx$
$= \frac{1}{6}\int_4^7 u^3 du$

$= \frac{1}{6} \cdot \frac{1}{4} [u^4]_4^7 = \frac{1}{24}(7^4 - 4^4)$
$= \frac{1}{24}(2401 - 256)$
$= \frac{2145}{24}$
$= \frac{715}{8}$

⑥ $\displaystyle\int_{\pi/3}^{\pi/2} \sin\theta \, d\theta = \left[-\cos\theta\right]_{\pi/3}^{\pi/2}$

$$= -\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\right)$$

$$= -\left(0 - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

⑦ $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2} = \left[\mathrm{Bgtan}\,x\right]_{-1}^{1}$

$$= \mathrm{Bgtan}\,1 - \mathrm{Bgtan}(-1)$$

$$= \frac{\pi}{4} - \frac{3\pi}{4}$$

$$= \frac{\pi}{2}$$

⑧ $\displaystyle\int_{-1}^{2} t \sin\frac{\pi t^2}{4} \, dt$ stel $u = \dfrac{\pi t^2}{4}$

$du = \dfrac{\pi}{2} t \, dt$

$$= \frac{2}{\pi}\int_{\pi/4}^{\pi} \sin u \, du = \frac{2}{\pi}\left[-\cos u\right]_{\pi/4}^{\pi}$$

$$= -\frac{2}{\pi}\left[\cos u\right]_{\pi/4}^{\pi}$$

$$= -\frac{2}{\pi}\left(\cos\pi - \cos\frac{\pi}{4}\right)$$

$$= -\frac{2}{\pi}\left(-1 - \frac{\sqrt{2}}{2}\right)$$

$$= \frac{2}{\pi} + \frac{\sqrt{2}}{\pi}$$

⑨ $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^3 x \, dx$ stel $u = \cos x$

$du = -\sin x \, dx$

$$= -\int_{0}^{0} (1-u^2) \, du$$

$$= \left(\int_0^0 du - \int_0^0 u^2 du\right) = -\left(\left[u\right]_0^0 - \left[\frac{u^3}{3}\right]_0^0\right)$$

$$= 0$$

(10) $\displaystyle\int_0^{2\tan\alpha} \frac{dx}{4+x^2} \quad (\alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[)$

$$= \frac{1}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4}\left[2\ \mathrm{Bgtan}\left(\frac{x}{2}\right)\right]_0^{2\tan\alpha}$$

$$= \frac{1}{4}(2\alpha - 0)$$

$$= \frac{\alpha}{2}$$

(11) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$\rightarrow$ eerst $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

$$= \int \frac{dx}{\sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

stel $t = \frac{x}{2}$

$dt = \frac{1}{2}dx$

$$= \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \mathrm{Bgsin}\, t + C$$

$$= \mathrm{Bgsin}\, \frac{x}{2} + C$$

$$= \left[\mathrm{Bgsin}\, \frac{x}{2}\right]_0^1$$

$$= \mathrm{Bgsin}\, \frac{1}{2} - \mathrm{Bgsin}\, 0$$

$$= \frac{\pi}{6} - \frac{\pi}{2}$$

$$= -\frac{2\pi}{6}$$

$$= -\frac{\pi}{3}$$

(12) $\displaystyle\int_{-1}^{+1} \frac{dx}{x^2+2x+5} = \int_{-1}^{+1} \frac{dx}{x^2+2x+1+4} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$

$$= \int_{-1}^{1} \frac{dx}{4\left(\left(\frac{x+1}{2}\right)^2+1\right)}$$

stel $u = \frac{x+1}{2}$

$du = \frac{1}{2}dx$

$$= \frac{1}{2}\int_0^1 \frac{dx}{u^2+1} = \frac{1}{2}\left[\mathrm{Bgtan}\, u\right]_0^1$$

$$= \frac{1}{2}(\mathrm{Bgtan}\, 1 - \mathrm{Bgtan}\, 0)$$

$$= \frac{\pi}{8}$$

(13)
$$\int_{-1}^{3} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{3} x\,dx$$
$$= -\frac{1}{2}\left[x^2\right]_{-1}^{0} + \frac{1}{2}\left[x^2\right]_{0}^{3}$$
$$= -\frac{1}{2}(0-1) + \frac{1}{2}(9-0)$$
$$= \frac{1}{2} + \frac{9}{2}$$
$$= 5$$

(14)
$$\int_{-\pi}^{\pi/2} \sin|x|\,dx = \int_{-\pi}^{0} \sin(-x)\,dx + \int_{0}^{\pi/2} \sin x\,dx$$
$$= -\int_{+\pi}^{0} \sin u\,du + \left[-\cos x\right]_{0}^{\pi/2}$$
$$= -\left[-\cos u\right]_{\pi}^{0} - \left[\cos x\right]_{0}^{\pi/2}$$
$$= \left[\cos u\right]_{\pi}^{0} - \left(\cos\frac{\pi}{2} - \cos 0\right)$$
$$= \cos 0 - \cos\pi - \cos\frac{\pi}{2} - \cos 0$$
$$= 1 - (-1) - (0-1)$$
$$= 1 - (-1) - (-1)$$
$$= 3$$

(15)
$$\int_{-1}^{2} \sqrt{x^2+x^4}\,dx = \int_{-1}^{2} \sqrt{x^2(1+x^2)}\,dx = \int_{-1}^{2} |x|\sqrt{1+x^2}\,dx$$
$$= \int_{-1}^{0} -x\sqrt{1+x^2}\,dx + \int_{0}^{2} x\sqrt{1+x^2}\,dx \qquad \text{stel } u = 1+x^2$$
$$du = 2x\,dx$$
$$= -\frac{1}{2}\int_{2}^{1} \sqrt{u}\,du + \frac{1}{2}\int_{1}^{5} \sqrt{u}\,du$$
$$= -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{2}^{1} + \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{1}^{5}$$
$$= -\frac{1}{3}\left[u^{3/2}\right]_{2}^{1} + \frac{1}{3}\left[u^{3/2}\right]_{1}^{5}$$
$$= -\frac{1}{3}\left(1-\sqrt{2^3}\right) + \frac{1}{3}\left(\sqrt{5^3}-1\right)$$

(16) $$\int_{\tan\alpha}^{\tan\beta} \frac{dx}{1+x^2} = \left[\operatorname{Arctan} x\right]_{\tan\alpha}^{\tan\beta}$$
$$= \beta - \alpha$$

(17) $$\int_{-1}^{1} (x^2+1)^2 dx = \int_{-1}^{1} x^4 dx + \int_{-1}^{1} 2x^2 dx + \int_{-1}^{1} dx$$
$$= \left[\frac{x^5}{5}\right]_{-1}^{1} + \left[\frac{2x^3}{3}\right]_{-1}^{1} + \left[x\right]_{-1}^{1}$$
$$= \frac{1}{5}(1+1) + \frac{2}{3}(1+1) + (1+1)$$
$$= \frac{2}{5} + \frac{4}{3} + 2$$
$$= \frac{6}{15} + \frac{20}{15} + \frac{30}{15}$$
$$= \frac{56}{15}$$

(18) $$\int_{t}^{2t} (t+z)\,dz = t\int_{t}^{2t} dz + \int_{t}^{2t} z\,dz$$
$$= t\left[z\right]_{t}^{2t} + \frac{1}{2}\left[z^2\right]_{t}^{2t}$$
$$= t(2t-t) + \frac{1}{2}(4t^2 - t^2)$$
$$= t^2 + \frac{3}{2}t^2$$
$$= \frac{5}{2}t^2$$

(19) $$\int_{2}^{1} \frac{x^2-1}{x^4}\,dx = \int_{2}^{1} \frac{1}{x^2}\,dx - \int_{2}^{1} \frac{1}{x^4}\,dx$$
$$= -\left[x^{-1}\right]_{2}^{1} + \frac{1}{3}\left[x^{-3}\right]_{2}^{1}$$
$$= -\left(1 - \frac{1}{2}\right) + \frac{1}{3}\left(1 - \frac{1}{8}\right)$$
$$= -\frac{12}{24} + \frac{7}{24}$$
$$= -\frac{5}{24}$$

(20) $$\int_3^6 \left(x^{-2}+\frac{1}{x^{-2}}\right)dx = \int_3^6 x^{-2}dx + \int_3^6 x^2 dx$$
$$= -\left[x^{-1}\right]_3^6 + \frac{1}{3}\left[x^3\right]_3^6$$
$$= -\left(\frac{1}{6}-\frac{1}{3}\right) + \frac{1}{3}(216-27)$$
$$= \frac{1}{6} + \frac{189}{3}$$
$$= 63 + \frac{1}{6}$$

(21) $$\int_{-1/3}^{2} \frac{dx}{(2x+1)^3}$$ soit $u = 2x+1$
$du = 2dx$

$$= \frac{1}{2}\int_{1/3}^{5} \frac{du}{u^3} = \frac{1}{2}\int_{1/3}^{5} u^{-3}du$$
$$= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left[u^{-2}\right]_{1/3}^{5}$$
$$= -\frac{1}{4}\left(\frac{1}{5^2} - \frac{1}{\left(\frac{1}{3}\right)^2}\right)$$
$$= -\frac{1}{4}\left(\frac{1}{125} - 9\right)$$
$$= \frac{1124}{500} = \frac{562}{250}$$

(22) $$\int_1^5 \left(\sqrt{x} - \frac{1}{\sqrt[3]{x}}\right)dx = \int_1^5 x^{1/2}dx - \int_1^5 x^{-1/3}dx$$
$$= \frac{2}{3}\left[x^{3/2}\right]_1^5 - \frac{3}{2}\left[x^{2/3}\right]_1^5$$
$$= \frac{2}{3}\left(5^{3/2}-1\right) - \frac{3}{2}\left(5^{2/3}-1\right)$$
$$= \frac{10\sqrt{5}}{3} - \frac{3\sqrt[3]{25}}{2} + \frac{5}{6}$$

(23) $$\int_5^1 \frac{t+1}{\sqrt{t}(\sqrt{t}+1)}dt$$

(23) $\displaystyle\int_5^1 \frac{t+1}{\sqrt{t}(\sqrt{t}+1)}\,dt$

Stel $u = \sqrt{t}+1 \to 2du = \frac{dt}{\sqrt{t}}$

$du = \frac{1}{2\sqrt{t}}\,dt$

$\sqrt{t} = u-1 \to t = (u-1)^2$

$$= 2\int_{1+\sqrt{5}}^{2} \frac{(u-1)^2+1}{u}\,du$$

$$= 2\int_{1+\sqrt{5}}^{2} \frac{u^2-2u+2}{u}\,du$$

$$= \left[u^2-4u+4\ln|u|\right]_{1+\sqrt{5}}^{2}$$

$$= 2^2-4\cdot 2+4\ln|2| - \left[(1+\sqrt{5})^2-4(1+\sqrt{5})+4\ln|1+\sqrt{5}|\right]$$

$$= -4+4\ln 2-(1+\sqrt{5})^2+4(1+\sqrt{5})-4\ln(1+\sqrt{5})$$

$$= -4+4\ln 2-(1+2\sqrt{5}+5)+4+4\sqrt{5}-4\ln(1+\sqrt{5})$$

$$= -6+2\sqrt{5}+4\ln 2-4\ln(1+\sqrt{5})$$

(24) $\displaystyle\int_0^1 \frac{x^2}{(4x+1)^{5/2}}\,dx$

(25) $\displaystyle\int_a^{2a} x\,\sqrt[3]{a+x}\,dx$

$u = a+x \to x = u-a$

$du = dx$

$$= \int_{2a}^{3a} (u-a)\sqrt[3]{u}\,du$$

$$= \int_{2a}^{3a} u^{4/3}\,du - a\int_{2a}^{3a} u^{1/3}\,du = \left[\frac{3}{7}u^{7/3}\right]_{2a}^{3a} - a\left[\frac{3}{4}u^{4/3}\right]_{2a}^{3a}$$

$$= \left(\frac{3}{7}(3a)^{7/3}-\frac{3}{7}(2a)^{7/3}\right)-a\left(\frac{3}{4}(3a)^{4/3}-\frac{3}{4}(2a)^{4/3}\right)$$

$$= 3a^{7/3}\left(\frac{\sqrt[3]{2187}}{7}-\frac{\sqrt[3]{128}}{7}-\frac{\sqrt[3]{81}}{4}+\frac{\sqrt[3]{16}}{4}\right)$$

$$= 3a^{7/3}\left(\frac{9\sqrt[3]{3}}{7}-\frac{4\sqrt[3]{2}}{7}-\frac{3\sqrt[3]{3}}{4}+\frac{2\sqrt[3]{2}}{4}\right)$$

(26) $\int_{16}^{25} \frac{5x+9}{(x-9)^2}$ stel $u = x-9 \Rightarrow x = u+9$
$du = dx$

$= \int_{16}^{25} \frac{5x}{(x-9)^2}dx + \int_{16}^{25} \frac{9}{(x-9)^2}dx$

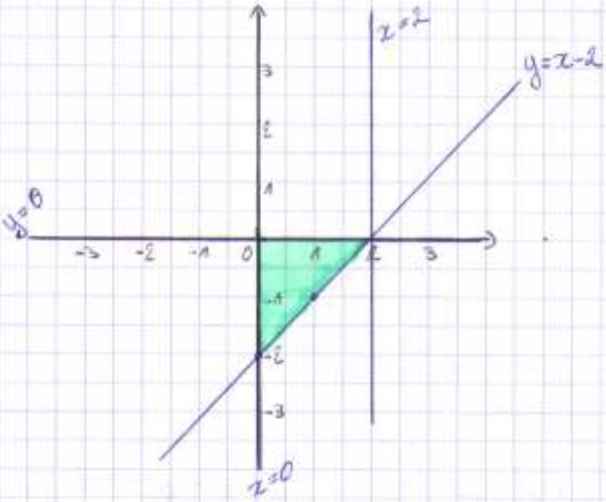
$= \int_{7}^{16} \frac{5}{u}du + \int_{7}^{16} \frac{45+9}{u^2}du = \left[5\ln|u|\right]_7^{16} + \left[-54u^{-1}\right]_7^{16}$

$= (5\ln|16| - 5\ln|7|) + \left(-\frac{54}{16} + \frac{54}{7}\right)$

$= 5\ln\left|\frac{16}{7}\right| + \left(\frac{-54\cdot 7 + 54\cdot 16}{112}\right)$

$= 5\ln\frac{16}{7} + \frac{486}{112} \quad = 5\ln\frac{16}{7} + \frac{243}{56}$

(2.2) (1) $y = x-2$; $y = 0$; $x = 0$; $x = 2$



• nulpunt: $x-2 = 0 \Rightarrow x = 2$

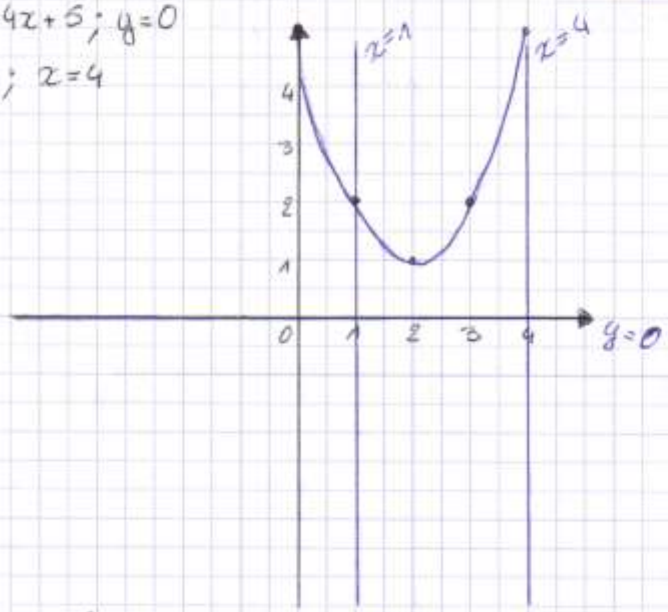
$\int_0^2 -(x-2)dx + \int_2^2 (x-2)dx$

$= -\left[\frac{x^2}{2} - 2x\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^2$

$= -(-2-0) + 0$

$= 2$

② $y=x^2-4x+5$; $y=0$
$x=1$; $x=4$



top : $x=-\frac{b}{2a}=2$ → $y=2^2-4\cdot 2+5=1$
→ min(2;1)

$$\int_1^4 (x^2-4x+5)\,dx = \int_1^4 x^2\,dx - \int_1^4 4x\,dx + \int_1^4 5\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^4 - \left[\frac{x^2}{2}\right]_1^4 + \left[5x\right]_1^4$$

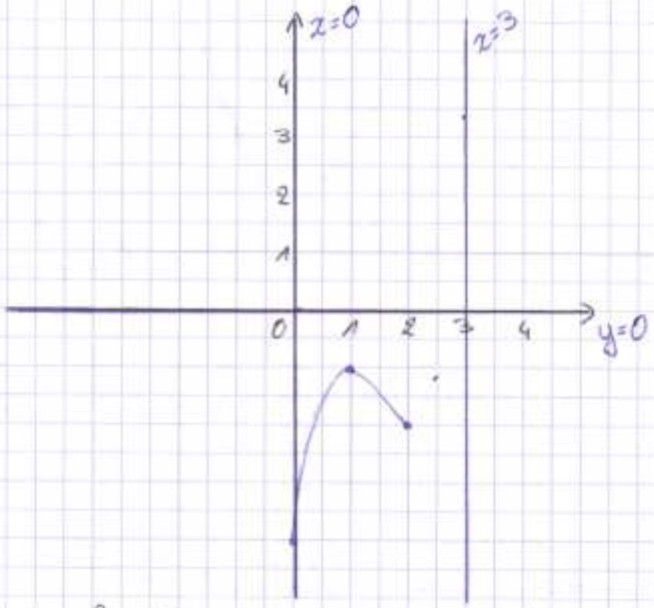
$$= \left(\frac{64}{3}-\frac{1}{3}\right) - 4\left(\frac{16}{2}-\frac{1}{2}\right) + (20-5)$$

$$= \frac{63}{3} - \frac{60}{2} + 15$$

$$= 21 - 30 + 15$$

$$= 6$$

③ $y = -2x^2 + 5x - 4$; $y = 0$; $x = 0$; $x = 3$



$$top = -\frac{b}{2a} = \frac{5}{4} \rightarrow y = -2\left(\frac{5}{4}\right)^2 + 5\cdot\left(\frac{5}{4}\right) - 4$$
$$= -\frac{7}{8}$$
$$\rightarrow min\left(\frac{5}{4}, -\frac{7}{8}\right)$$

$$\int_0^3 (-2x^2 + 5x - 4)\,dx = \int_0^3 -2x^2\,dx + \int_0^3 5x\,dx - \int_0^3 4\,dx$$
$$= -2\left[\frac{x^3}{3}\right]_0^3 + 5\left[\frac{x^2}{2}\right]_0^3 - 4\left[x\right]_0^3$$
$$= -2.9 + \frac{45}{2} - 12$$
$$= -\frac{36}{2} + \frac{45}{2} - \frac{24}{2}$$
$$= (-)\,\frac{15}{2}$$

(4) $y = 3x+7$; $y=0$; $x=1$; $x=-3$

nulpunt: $3x+7=0 \rightarrow x = -\frac{7}{3}$

$$\int_{-3}^{-7/3} -(3x+7)dx + \int_{-7/3}^{1} (3x+7)dx = -\left[\frac{3}{2}x^2+7x\right]_{-3}^{-7/3} + \left[\frac{3}{2}x^2+7x\right]_{-7/3}^{1}$$

$$= -\left(\frac{3}{2}\cdot\left(-\frac{7}{3}\right)^2 + 7\left(-\frac{7}{3}\right) - \frac{3}{2}\cdot(-3)^2 + 7(-3)\right) + \left(\frac{3}{2} + 7 - \frac{3}{2}\left(-\frac{7}{3}\right)^2 + 7\left(-\frac{7}{3}\right)\right)$$

$$= -\left(\frac{3}{2}\cdot\frac{49}{9} - \frac{49}{3} - \frac{27}{2} - 21\right) + \left(\frac{3}{2} + 7 - \frac{3}{2}\cdot\frac{49}{9} - \frac{49}{3}\right)$$

$$= -\frac{49}{6} + \frac{49}{3} + \frac{27}{2} + \ldots$$

(2.3) (1) $$\int_2^4 \left(\frac{x^2-4}{x^2} - 0\right)dx = \int_2^4 dx - 4\int_2^4 \frac{1}{x^2}dx$$

$$= [x]_2^4 + 4\left[\frac{1}{x}\right]_2^4$$

$$= (4-2) + 4\left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= 2 - 4\cdot\frac{1}{4}$$

$$= 1$$

(2) Snijpunten: $\begin{cases} y = x^2 \\ y = x \end{cases} \Leftrightarrow x^2 = x$

$\Leftrightarrow x^2 - x = 0$

$\Leftrightarrow x(x-1) = 0$ $\nearrow (0,0)$ $\searrow (1,1)$

$\begin{cases} y = x^2 \\ y = 2x \end{cases} \Leftrightarrow x^2 = 2x$

$\Leftrightarrow x^2 - 2x = 0$

$\Leftrightarrow x(x-2) = 0$ $\nearrow (2,4)$ $\searrow (0,0)$

$$\int_0^1 (2x-x)\,dx + \int_1^2 (2x-x^2)\,dx = \int_0^1 x\,dx + \int_1^2 (2x-x^2)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2} + \left(4 - \frac{8}{3} - 1 + \frac{1}{3}\right)$$

$$= \frac{1}{2} + \left(3 - \frac{7}{3}\right)$$

$$= \frac{7}{6}$$

3

(4) $y^2=2x$ en $y=x-4$

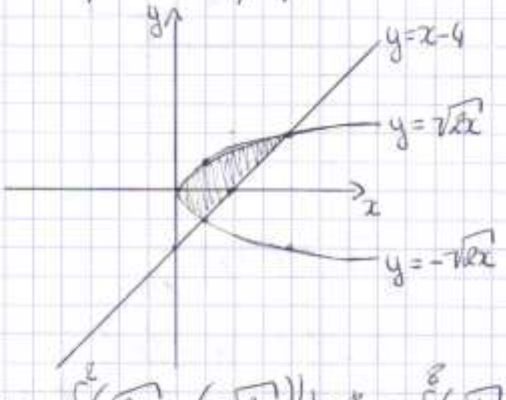
Snijpunten:

$$\begin{cases} y^2=2x \\ y=x-4 \end{cases} \leftrightarrow \begin{cases} (x-4)^2=2x \\ y=x-4 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2-8x+16-2x=0 \\ y=x-4 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2-10x+16=0 \\ y=x-4 \end{cases} \qquad D=100-64=36 \qquad x_{12}=\frac{10\pm 6}{2}$$

$\rightarrow (8,4)$ en $(2,-2)$



Oppervlakte: $\int_0^2 (\sqrt{2x}-(-\sqrt{2x}))dx + \int_2^8 (\sqrt{2x}-(x-4))dx$

$$= \int_0^2 2\sqrt{2x}\,dx + \int_2^8 (\sqrt{2x}-x+4)dx$$

Beter en sneller: integratie volgens y

$$\int_{y=-2}^{y=4} \left((y+4)-\frac{y^2}{2}\right)dy = \left[\frac{y^2}{2}+4y-\frac{y^3}{6}\right]_{-2}^{4}$$

$$= \left(8+16-\frac{64}{6}\right)-\left(2-8+\frac{8}{6}\right)$$

$$= 24-\frac{64}{6}+6-\frac{8}{6}$$

$$= 30-12$$

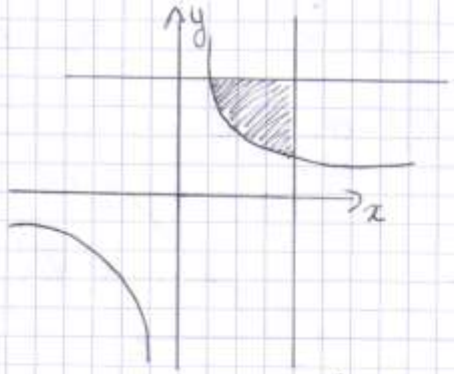
$$= 18$$

⑤ $y = \frac{1}{x}$; $y = 2$; $x = 2$

Snijpunten :

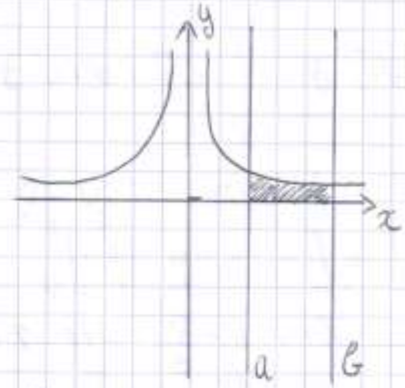
$\begin{cases} y = \frac{1}{x} \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = 2 \end{cases} \rightarrow \left(\frac{1}{2}, 2\right)$

$\begin{cases} y = \frac{1}{x} \\ x = 2 \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{2} \\ x = 2 \end{cases} \rightarrow \left(2, \frac{1}{2}\right)$ en $(2, 2)$



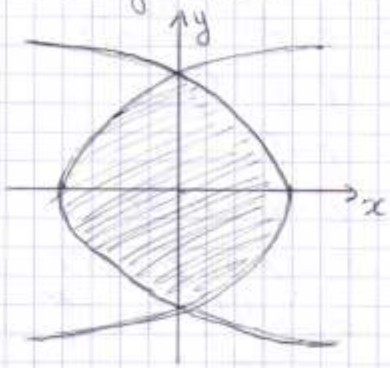
$$\int_{1/2}^{2} \left(2 - \frac{1}{x}\right) dx = \left[2x - \ln|x|\right]_{1/2}^{2}$$
$$= 4 - \ln 2 - 1 + \ln \frac{1}{2}$$
$$= 4 - \ln 2 - 1 + \ln 1 - \ln 2$$

⑥



$$= \int_{a}^{b} \left(\frac{1}{x^2} - 0\right) dx$$
$$= \left[-\frac{1}{x}\right]_{a}^{b} = -\frac{1}{b} + \frac{1}{a}$$
$$= \frac{1}{a} - \frac{1}{b}$$

⑦ $y^2 = x+1$ en $y^2 = -x+1$



Integratie volgens de y-as:

$$\int_{-1}^{1}\left[(1-y^2)-(y^2-1)\right]dy = \int_{-1}^{1}(-2y^2+2)dy$$

$$= \left[-\frac{2}{3}y^3+2y\right]_{-1}^{1}$$

$$= -\frac{2}{3}+2-\left(\frac{2}{3}-2\right)$$

$$= 4-\frac{4}{3}$$

$$= \frac{8}{3}$$

(2.4) ① $p = 0{,}09q^2$ (aanbod) ; $p = 300-0{,}03q^2$ (vraag)

Bepalen van het marktevenwicht:

$$\begin{cases} p = 0{,}09q^2 \\ p = 300-0{,}3q^2 \end{cases} \Leftrightarrow 0{,}09q^2 = 300-0{,}03q^2$$

$$\Leftrightarrow 0{,}12q^2 = 300$$

$$\Leftrightarrow q^2 = 2500$$

$$\Leftrightarrow q = \pm 50 \rightarrow p = 0{,}09.(50)^2 = 225$$

$$CS = \int_0^{50}(300-0{,}03q^2-225)dq = \int_0^{50}(75-0{,}03q^2)dq$$

$$= \left[75q-0{,}01q^3\right]_0^{50}$$

$$= 3750-1250 = 2500$$

$$PS = \int_0^{50}(225-0{,}09q^2)dq = \left[225q-0{,}03q^3\right]_0^{50}$$

$$= 225.50-0{,}03.(50)^3$$

$$= 11250-3750 = 7500$$

(2) $q = p-1$ (aanbod) en $q = \frac{60}{p} - 5$ (vraag)
Bepalen van het marktevenwicht:

$$\begin{cases} q = p-1 \\ q = \frac{60}{p} - 5 \end{cases} \Leftrightarrow p-1 = \frac{60}{p} - 5$$
$$\Leftrightarrow p - \frac{60}{p} = -4$$
$$\Leftrightarrow p^2 + 4p - 60 = 0$$
$$D = 16 + 4.60 = 256$$
$$x_{12} = \frac{4 \pm 16}{2} = \begin{cases} \cancel{-10} \\ 6 \rightarrow q = 5 \end{cases}$$

$$CS: \int_0^5 \left(\frac{60}{q+5} - 6\right) dq = \int_0^5 \frac{60}{q+5} dq - [6q]_0^5 \qquad \text{stel } u = q+5, \; du = dq$$
$$= 60 \int_5^{10} \frac{1}{u} du - [6q]_0^5 = [60 \ln u]_5^{10} - [6q]_0^5$$
$$= 60(\ln 10 - \ln 5) - 30$$
$$= 60 \ln 2 - 30$$

$$PS: \int_0^5 (6 - (q+1)) dq = \int_0^5 (5-q) dq$$
$$= \left[5q - \frac{1}{2} q^2\right]_0^5$$
$$= \frac{25}{2}$$

(2.5) (1) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} dx$ is continu in $[0, \frac{\pi}{2}[$ en disc in $\frac{\pi}{2}$

$$\rightarrow \lim_{t \xrightarrow{<} \frac{\pi}{2}} \int_0^t \frac{\cos x}{\sqrt{1 - \sin x}}$$

Stap 1: $\int_0^t \frac{\cos x}{\sqrt{1-\sin x}} dx$ stel $u = 1 - \sin x$, $du = -\cos x \, dx$

$$= -\int_1^{1-\sin t} \frac{du}{\sqrt{u}} = -\int_1^{1-\sin t} u^{-1/2} du = -\left[2u^{1/2}\right]_1^{1-\sin t}$$
$$= -\left(2\sqrt{1-\sin t} - 2\right)$$
$$= -2\left(\sqrt{1-\sin t} - 1\right)$$

Stap 2: $\lim_{t \xrightarrow{<} \frac{\pi}{2}} (-2\sqrt{1-\sin t} + 2) = -2\sqrt{1-1} + 2$
$$= 2$$

② $\int_0^{+\infty} \frac{dx}{x^2+4} \to \lim_{x\to+\infty} \int_0^t \frac{dx}{x^2+4}$

$= \frac{1}{4}\int_0^t \frac{dx}{\left(\frac{x}{2}\right)^2+1}$ stel $u = \frac{x}{2}$
$du = \frac{1}{2}dx$

$= \frac{1}{2}\int_0^t \frac{du}{u^2+1}$

$= \frac{1}{2}\left[\text{Bgtan}\, t\right]_0^{t/2} = \frac{1}{2}\left(\text{Bgtan}\frac{t}{2} - \text{Bgtan}\,0\right)$

$= \frac{1}{2}\,\text{Bgtan}\frac{t}{2}$

$\lim_{t\to+\infty} \frac{1}{2}\text{Bgtan}\frac{t}{2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

③ $\int_0^{+\infty} \cos x\, dx \to \lim_{x\to+\infty} \int_0^t \cos x\, dx$

$= \left[\sin x\right]_0^t$

$= \sin t - \sin 0$

$= \sin t$

$\lim_{t\to+\infty} \sin t \to$ bestaat niet

④ $\int_0^1 \frac{dx}{\sqrt{1-x^2}} \to \lim_{t\overset{<}{\to}1} \int_0^t \frac{dx}{\sqrt{1-x^2}}$

$= \left[\text{Bgsin}\, x\right]_0^t = \text{Bgsin}\, t$

$\lim_{t\to1} \text{Bgsin}\, t = \frac{\pi}{2}$

⑤ $\int_0^1 \frac{dx}{x} = \lim_{t\overset{>}{\to}0} \int_t^1 \frac{dx}{x}$

$= \left[\ln x\right]_t^1$

$= \ln 1 - \ln t$

$= 0 - (-\infty) = +\infty$

(6) $\displaystyle\int_0^{+\infty} e^{-x} \rightarrow \lim_{t \to +\infty} \int_0^t e^{-x} dx$

$$= \int_0^t e^{-x} dx = \left[-e^{-x}\right]_0^t$$

$$= -e^{-t} + e^0$$

$$= -e^{-t} + 1$$

$$\lim_{t \to +\infty} (-e^{-t} + 1) = -e^{-\infty} + 1$$

$$= \frac{1}{+\infty} + 1$$

$$= 1$$

(7) $\displaystyle\int_{-\infty}^{+\infty} x \cdot e^{-x^2}$

(2.6) $\int_a^{+\infty} \frac{dx}{x^m} \quad (a>0) \rightarrow \lim_{t\to+\infty} \int_a^t \frac{dx}{x^m}$

1e geval: $m=1$

$$\rightarrow \lim_{t\to+\infty} \int_a^t \frac{dx}{x} = \lim_{t\to+\infty} [\ln|x|]_a^t$$

$$= \lim_{t\to+\infty} (\ln t - \ln a)$$

$$= +\infty - \ln a = +\infty$$

↳ divergent

2e geval: $m \neq 1$

$$\rightarrow \lim_{t\to+\infty} \left[\frac{1}{-(m+1)} x^{-(m+1)}\right]_a^t = \lim_{t\to+\infty} \left(\frac{t^{-m+1}}{-m+1} - \frac{a^{-m+1}}{-m+1}\right)$$

(2a) $-m+1>0 \Leftrightarrow m<1$

$$\Rightarrow +\infty - \frac{a^{-m+1}}{-m+1} = +\infty$$

↳ divergent

(2b) $-m+1<0 \Leftrightarrow m>1$

$$\Rightarrow 0 - \frac{a^{-m+1}}{-m+1} = -\frac{a^{-m+1}}{-m+1}$$

(2.7) $\int_0^a \frac{dx}{\sqrt{x^m}} \quad (a>0) \rightarrow \lim_{t\overset{>}{\to}0} \int_t^a x^{-m/2}\,dx$

1e geval: $\frac{m}{2}=1 \Leftrightarrow m=2$

$$\rightarrow \lim_{t\overset{>}{\to}0} \int_t^a \frac{dx}{x} = \lim_{t\overset{>}{\to}0} [\ln|x|]_t^a$$

$$= \lim_{t\overset{>}{\to}0} (\ln|a| - \ln|t|)$$

$$= \ln a - (-\infty) = +\infty$$

2e geval: $\frac{m}{2} \neq 1 \Leftrightarrow m \neq 2$

$$\rightarrow \lim_{t\overset{>}{\to}0} \int_t^a x^{-m/2}\,dx = \lim_{t\overset{>}{\to}0} \left[\frac{x^{-m/2+1}}{-\frac{m}{2}+1}\right]_t^a$$

$$= \lim_{t\overset{>}{\to}0} \left[\frac{a^{-m/2+1}}{-\frac{m}{2}+1} - \frac{t^{-m/2+1}}{-\frac{m}{2}+1}\right]$$

$$= \frac{a^{-m/2+1}}{-m/2+1} - \frac{\lim_{t\overset{>}{\to}0} t^{-m/2+1}}{-m/2+1}$$

(2a) $-\frac{m}{2}+1>0 \Leftrightarrow m<2$

(zelf zie minerva)

(2.8) (1) $\displaystyle\int_{10}^{15}\frac{1}{5}e^{-(1/5)x}\,dx = \left[-e^{-1/5x}\right]_{10}^{15}$

$= -e^{-3}+e^{-2}$

(2) $\displaystyle\int_{0}^{8}\frac{1}{5}e^{-1/5x}\,dx = \left[-e^{-1/5x}\right]_{0}^{8}$

$= -e^{8/5}+1$

(3) $\displaystyle\int_{12}^{+\infty}\frac{1}{5}e^{-1/5x}\,dx \to \lim_{t\to+\infty}\int_{12}^{t}\frac{1}{5}e^{-1/5x}\,dx$

$= \left[-e^{-1/5x}\right]_{12}^{t}$

$= -e^{-t/5}+e^{-12/5}$

$\displaystyle\lim_{t\to+\infty}\left(-e^{-t/5}+e^{-12/5}\right) = 0+e^{-12/5}$

(4) $\displaystyle\int_{0}^{+\infty}\frac{1}{5}e^{-1/5x}\,dx \to \lim_{t\to+\infty}\int_{0}^{t}\frac{1}{5}e^{-1/5x}$

$= \left[-e^{-1/5x}\right]_{0}^{t}$

$= -e^{-1/5t}+1$

$\displaystyle\lim_{t\to+\infty}\left(-e^{-1/5t}+1\right) = 0+1$

$= 1$

(2.9) $f(x)=\frac{1}{4}e^{-x/4}$ (x = wachttijd in min)

$P(x\geq T)=\displaystyle\int_{T}^{+\infty}\frac{1}{4}e^{-x/4}\,dx$

$\displaystyle\lim_{a\to+\infty}\int_{T}^{a}\frac{1}{4}e^{-x/4}\,dx = \lim_{a\to+\infty}\left[-e^{-x/4}\right]_{T}^{a}$

$= -e^{-a/4}+e^{-T/4}$

$= -e^{-\infty}+e^{-T/4}$

$= 0+e^{-T/4}$

Hoofdstuk 3: Differentiaalvergelijkingen van orde één

(3.1) ① $(1+x^2)y' + xy = 0$

$\Leftrightarrow (1+x^2)\frac{dy}{dx} = -xy$

$\Leftrightarrow \int \frac{dy}{y} = \int \frac{-x}{1+x^2}\,dx$

Is $y=0$ een oplossing vd DV?
$(1+x^2).0 + x.0 = 0$
→ $y=0$ is een oplossing!

$\Leftrightarrow \ln|y| = -\frac{1}{2}\int \frac{du}{u}$ stel $u = 1+x^2$, $du = 2x\,dx$

$\Leftrightarrow \ln|y| = -\frac{1}{2}\ln|1+x^2| + C$

→ impliciete algemene oplossing

$\Leftrightarrow e^{\ln|y|} = e^{\ln\frac{1}{\sqrt{1+x^2}} + C}$

$\Leftrightarrow |y| = e^C \cdot \frac{1}{\sqrt{1+x^2}}$ $\underbrace{(e^C \in \mathbb{R}_0^+)}_{c_1}$

$\Leftrightarrow y = \frac{\pm c_1}{\sqrt{1+x^2}}$

② $xy\,dx + (x-y)dy = 0$

$\Leftrightarrow (x-y)\,dy = -xy\,dx$

$\Leftrightarrow \frac{(x-y)dy}{y} = -x\,dx$

→ kunnen niet gescheiden worden

③ $y' + 2y = 3x - 2$

$\Leftrightarrow \frac{dy}{dx} + 2y = 3x - 2$

→ kunnen niet gescheiden worden

④ $y^2dx - (1-x)\,dy = 0$

Is $y=0$ een oplossing vd DV?
$0.dx - (1-x).0' = 0$
→ $y=0$ is een oplossing

$\Leftrightarrow y^2dx = (1-x)dy$

$\Leftrightarrow \int \frac{dx}{1-x} = \int \frac{dy}{y^2}$ $u = 1-x$, $du = -dx$

$\Leftrightarrow -\int \frac{du}{u} = \int y^{-2}dy$

$\Leftrightarrow -\ln|1-x| = -\frac{1}{y}$ $\Leftrightarrow \frac{1}{y} = \ln|1-x| + C$

$\Leftrightarrow y = \dfrac{1}{\ln|1-x|+C}$

(5) $y' = \sqrt{1-y^2}$

$\Leftrightarrow \dfrac{dy}{dx} = \sqrt{1-y^2}$

$\Leftrightarrow \dfrac{1}{\sqrt{1-y^2}}\,dy = dx$

Is $y=1$ of $y=-1$ een oplossing v/d DV?

$(\pm 1)' = \sqrt{1-1^2}$

$0=0 \rightarrow$ ja!

$\Leftrightarrow \displaystyle\int \frac{1}{\sqrt{1-y^2}}\,dy = \int dx$

$\Leftrightarrow \text{Bgsin}\,y = x + C$

$\Leftrightarrow y = \sin(x+C)$

(6) $\dfrac{dy}{dx} = \dfrac{\cos^2 y \tan y}{1+x^2}$

$\Leftrightarrow \displaystyle\int \frac{dy}{\cos^2 y \tan y} = \int \frac{dx}{1+x^2}$

$u = \tan y$

$du = \dfrac{1}{\cos^2 y}\,dy$

$\Leftrightarrow \displaystyle\int \frac{du}{u} = \text{Bgtan}\,x + C$

$\Leftrightarrow \ln|u| = \text{Bgtan}\,x + C$

$\Leftrightarrow \ln|\tan y| = \text{Bgtan}\,x + C$

$\Leftrightarrow e^{\ln|\tan y|} = e^{\text{Bgtan}\,x + C}$

$\Leftrightarrow \tan y = e^{C} \cdot e^{\text{Bgtan}\,x}$

$\Leftrightarrow y = \text{Bgtan}(C_1 \cdot e^{\text{Bgtan}\,x}) \qquad (C_1 \in \mathbb{R}_0^+)$

en $\tan y = 0$

(7) $2(y+3)\,dx + xy\,dy = 0$

$\Leftrightarrow 2(y+3)\,dx = -xy\,dy$

$\Leftrightarrow \dfrac{2\,dx}{-x} = \dfrac{y\,dy}{y+3}$

Is $y=-3$ een oplossing?

$2(-3+3)\,dx = x\cdot(-3)\cdot 0$

$\rightarrow 0=0$

$\rightarrow$ ja

$\Leftrightarrow -2\displaystyle\int \frac{dx}{x} = \int \frac{y}{y+3}\,dy$

$\Leftrightarrow -2\ln|x| = \displaystyle\int \frac{y+3-3}{y+3}\,dy$

$\Leftrightarrow -2\ln|x| = y - 3\displaystyle\int \frac{dy}{y+3}$

$\Leftrightarrow y - 3\ln|y+3| = -2\ln|x| + C \quad$ en $y=-3$

⑧ $\frac{dy}{dx} = e^{x+y}$

$\Leftrightarrow \frac{dy}{dx} = e^x . e^y$

$\Leftrightarrow \int \frac{dy}{e^y} = \int e^x dx$

$\Leftrightarrow e^{-y} = -e^x + C$

$\Leftrightarrow \ln(e^{-y}) = \ln(-e^x + C)$

$\Leftrightarrow y = -\ln(-e^x + C)$

⑨ $\frac{dy}{dx} + y . \cot x = 0$

$\Leftrightarrow -\int \frac{dy}{y} = \int \cot x \, dx$ → Is $y=0$ een oplossing v/d DV?

$0' + 0 . \cot x = 0$

$0 = 0$ → ja

$\Leftrightarrow -\ln|y| = \int \frac{\cos x}{\sin x} dx$

$\Leftrightarrow \ln|y| = -\ln|\sin x| + C$

$\Leftrightarrow e^{\ln|y|} = e^{\ln|\sin x^{-1}| + C}$

$\Leftrightarrow y = \frac{e^c}{\sin x}$ en $y = 0$

⑩ $dy = y^{-1} x^4 (y^2+1) dx$

$\Leftrightarrow \int \frac{y \, dy}{y^2+1} = \int x^4 dx$ stel $u = y^2+1$, $du = 2y\,dy$

$\Leftrightarrow \frac{1}{2} \ln|y^2+1| = \frac{x^5}{5} + C$

$\Leftrightarrow \ln|y^2+1| = \frac{2}{5} x^5 + C$

$\Leftrightarrow e^{\ln|y^2+1|} = e^{2/5 x^5} e^C$

$\Leftrightarrow y^2 = C . e^{2/5 x^2} - 1$ $(C \in \mathbb{R}_0^+)$

$\Leftrightarrow y = \pm\sqrt{C . e^{2/5 x^2} - 1}$

⑪ $y' = 3xy$

$\Leftrightarrow \int \frac{dy}{y} = \int 3x \, dx$

$\Leftrightarrow \ln y = \frac{3}{2} x^2 + C$

$\Leftrightarrow y = e^{3/2 x^2 + C}$

$\Leftrightarrow y = C . e^{3/2 x^2}$

(12) $(1+x^2)y' = y^2$ met $y = -\frac{4}{\pi}$ als $x = 1$

$\Leftrightarrow \int \frac{dy}{y^2} = \int \frac{dx}{1+x^2}$ → Is $y=0$ een oplossing vd DV?

$\Leftrightarrow -\frac{1}{y} = \text{Bgtan}x + C$ $\qquad (1+x^2)0' = 0^2$

$\Leftrightarrow y = -\frac{1}{\text{Bgtan}x + C}$ $\qquad \rightarrow$ ja

Invullen van de randvoorwaarde in de algemene explicete oplossing geeft ons:

$-\frac{4}{\pi} = -\frac{1}{\text{Bgtan}1 + C} \Rightarrow \frac{4}{\pi} = \frac{1}{\frac{\pi}{4} + C}$

$\Rightarrow C = 0$

Oplossing randvoorwaardeprobleem

$y = -\frac{1}{\text{Bgtan}x + 0} = -\frac{1}{\text{Bgtan}x}$

(3.2) (1) $dy + (x^5 - x^2y)dx = 0$

$\Leftrightarrow \frac{dy}{dx} = -(x^5 - x^2y)$

$\Leftrightarrow y' \underbrace{\boxed{-x^2}}_{P(x)} y = \underbrace{\boxed{-x^5}}_{Q(x)}$

Stap 1: $\Phi = \int P(x)dx = \int -x^2 dx = \frac{-x^3}{3}$

$\rightarrow G(x) = e^{\Phi(x)} = e^{-x^3/3}$

Stap 2: $\Phi = \int G(x)\, Q(x)\, dx$

$= \int e^{-x^3/3} (-x^5) dx$ $\qquad$ stel $u = -\frac{x^3}{3} \Leftrightarrow x^3 = -3u$

$du = -x^2 dx$

$= \int e^u (-3u) du$

$f(x) = -3u \qquad f'(x) = -3$

$g'(x) = e^u \qquad g(x) = e^u$

$= -3ue^u + 3\int e^u du = -3u \cdot e^u + 3 \cdot e^u$

$= -3 \cdot \frac{x^3}{3} \cdot e^{-x^3/3} + 3 \cdot e^{-x^3/3} = e^{-x^3/3}(x^3 + 3)$

<u>Stap 3</u>: $y = \frac{1}{G(x)}\left[\Psi(x) + C\right]$

$$= \frac{1}{e^{-x^3/3}}\left[e^{-x^3/3}(x+3) + C\right]$$

$$= x^3 + 3 + C\,e^{x^3/3}$$

(2) $(x-1)y' - y = (x-1)^3 \sin x$ met $y=0$ als $x=0$

$$\Leftrightarrow y' \underbrace{\boxed{-\frac{1}{x-1}}}_{P(x)} y = \underbrace{\boxed{(x-1)^2 \sin x}}_{Q(x)}$$

<u>Stap 1</u>: $\Phi = \int -\frac{1}{x-1}\,dx = -\int \frac{1}{x-1}\,dx$ stel $u = x-1$, $du = dx$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|x-1|$$

$$\rightarrow G(x) = e^{-\ln|x-1|} = e^{\ln|x-1|^{-1}} = \frac{1}{x-1}$$

<u>Stap 2</u>: $\Psi = \int \frac{1}{|x-1|}\,(x-1)^2 \sin x\,dx$

$$= \int \frac{1}{\pm(x-1)}\,(x-1)^2 \sin x\,dx$$

$$= \pm \int (x-1)\sin x\,dx$$

$$f(x) = (x-1) \qquad f'(x) = 1$$
$$g'(x) = \sin x \qquad g(x) = -\cos x$$

$$= \pm\left[(x-1)(-\cos x) + \int \cos x\,dx\right.$$

$$= \pm\left[-(x-1).\cos x + \sin x\right.$$

<u>Stap 3</u>: $y = \frac{1}{G(x)} \cdot \left[\Psi(x) + C\right]$

$$= |x-1|\left[\pm\left(-(x-1)\cos x + \sin x\right) + C\right]$$

$$= \pm(x-1)\left[\pm\left(-(x-1).\cos x + \sin x\right) + C\right]$$

$$= (x-1)^2.\cos x + (x-1).\sin x + C(x-1)$$

Randvoorwaarde: $y=0$ als $x=0$

$$0=-(0-1)^2\cdot\cos 0+(0-1)\sin 0+C(0-1)$$
$$=-1\cdot 1+(-1)\cdot 0-C$$
$$\Rightarrow C=-1$$

Particuliere oplossing: $y=-(x-1)^2\cos x+(x-1)\sin x-(x-1)$

③ $x^2dx+xy\,dy=0$

$\Leftrightarrow x^2dx=-xy\,dy$

$\Leftrightarrow x^2=-xy\cdot y'$

$\Leftrightarrow y'+\frac{x}{y}=0$

$\rightarrow$ geen differentiaalvergelijking van orde één

④ $\frac{dy}{dx}\underbrace{\boxed{-x}}_{P(x)}y=\underbrace{\boxed{x^3}}_{Q(x)}$

Stap 1: $\Phi=\int -x\,dx=-\int x\,dx=-\frac{1}{2}x^2$

$\rightarrow G(x)=e^{-1/2x^2}$

Stap 2: $\Psi=\int G(x)\cdot Q(x)dx$

$$=\int e^{-1/2x^2}x^3dx \qquad \text{stel } u=\tfrac{1}{2}x^2,\ du=x\,dx$$

$$=2\int e^{-u}\cdot u\,du$$

$$f(u)=u \qquad f'(u)=1$$
$$g'(u)=e^{-u} \qquad g(u)=-e^{-u}$$

$$=2\left(-u\,e^{-u}+\int e^{-u}du\right)$$
$$=-2\cdot\frac{1}{2}x^2e^{-1/2x^2}-2e^{-1/2x^2}+C$$
$$=-x^2e^{-1/2x^2}-2e^{-1/2x^2}+C$$

Stap 3: $y=\frac{1}{G(x)}\cdot[\Psi(x)+C]$

$$=-x^2-2+C\cdot e^{1/2x^2}$$

⑤ $xy' = y + x^3 \ln x$

$\Leftrightarrow y' = \frac{y}{x} + x^2 \ln x$

$\Leftrightarrow y' \underbrace{\boxed{-\frac{1}{x}}}_{P(x)} y = \underbrace{\boxed{x^2 \ln x}}_{Q(x)}$

__Stap 1:__ $\Phi = \int -\frac{1}{x} = -\ln|x|$

$\rightarrow G(x) = e^{-\ln|x|} = \frac{1}{x}$

__Stap 2:__ $\Psi = \int \frac{1}{x} \cdot x^2 \ln x \, dx = \int x \ln x \, dx$

$f(x) = \ln x \quad f'(x) = \frac{1}{x}$

$g'(x) = x \quad g(x) = \frac{1}{2} x^2$

$\Rightarrow \ln|x| \, \frac{1}{2} x^2 - \frac{1}{2} \int x \, dx$

$\Rightarrow \frac{1}{2} x^2 \ln|x| - \frac{1}{4} x^2 + C$

__Stap 3:__ $y = \frac{1}{G(x)} \cdot [\Psi(x) + C] = x\left(\frac{1}{2} x^2 \ln|x| - \frac{1}{4} x^2 + C\right)$

$= \frac{1}{2} x^3 \ln|x| - \frac{1}{4} x^3 + Cx$

⑥ $\frac{dy}{dx} + y \underbrace{\boxed{\cot x}}_{P(x)} = \underbrace{\boxed{5 \cdot e^{\cos x}}}_{Q(x)}$

__Stap 1:__ $\Phi = \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$ stel $u = \sin x$, $du = \cos x \, dx$

$= \int \frac{du}{u} = \ln|\sin x|$

$\rightarrow G(x) = \sin x$

__Stap 2:__ $\Psi(x) = \int \sin x \; 5 \cdot e^{\cos x} = \pm 5 \int \sin x \cdot e^{\cos x}$

$= \pm 5 \int e^u (-du)$ stel $u = \cos x$, $du = -\sin x \, dx$

$= \pm(-5e^u) = \mp 5 \cdot e^{\cos x}$

__Stap 3:__ $y = \frac{1}{\pm \sin x} \left[\pm(-5 \cdot e^{\cos x}) + C\right]$

$= \frac{1}{\sin x} \left[-5 \cdot e^{\cos x} + C\right]$

⑦ $xy' = y + x^3 + 3x^2 - 2x$

$\Leftrightarrow y' - \frac{1}{x} y = x^2 + 3x - 2$

<u>Stap 1:</u> $\int P(x) dx = -\ln x$

$\rightarrow G(x) = \frac{1}{x}$

<u>Stap 2:</u> $\int \frac{1}{x}(x^2 + 3x - 2) dx$

$\Leftrightarrow \int (x + 3 - \frac{2}{x}) dx$

$\Leftrightarrow \frac{1}{2}x^2 + 3x - 2\ln x + C$

<u>Stap 3:</u> $\frac{1}{1/x} \cdot (\frac{1}{2}x^2 + 3x - 2\ln x + C$

$= \frac{1}{2}x^3 + 3x^2 - 2x\ln x + Cx$

⑧ $dy + (2y - e^{-x}) dx = 0$

$\Leftrightarrow dy = -(2y - e^{-x}) dx$

$\Leftrightarrow y' + 2y = +e^{-x}$

<u>Stap 1:</u> $\int P(x) = \int 2 dx = 2x$

$\rightarrow G(x) = e^{2x}$

<u>Stap 2:</u> $\int e^{2x} e^{-x} dx = \int e^x dx = e^x + C$

<u>Stap 3:</u> $\frac{1}{e^{2x}} \cdot [e^x + C]$

$= e^{-x} + C e^{-2x}$

(3.3) (1) $y' + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4$ $\qquad p = 4$

<u>Stap 1</u>: $y^{-4}y' + \frac{1}{3}y^{-3} = \frac{1}{3}(1-2x)$

<u>Stap 2</u>: $y^4 = 0$ een oplossing vd DV

$\rightarrow 0' + \frac{1}{3}0 = \frac{1}{3}(1-2x).0^4$

$\Leftrightarrow 0 = 0 \rightarrow$ ja

<u>Stap 3</u>: stel $z = \frac{1}{1-p}y^{1-p} = \frac{1}{1-4}y^{1-4} = -\frac{1}{3}y^{-3}$

$\rightarrow z' = y^{-p}.y' = y^{-4}y'$

$\rightarrow z' + (1-p)R(x)z = S(x)$

$\Leftrightarrow z' + (1-4)\frac{1}{3}z = \frac{1}{3}(1-2x)$

$\Leftrightarrow z' - z = \frac{1}{3}(1-2x)$

<u>Stap 4</u>: $\Phi(x) = \int - dx = -x$

$\rightarrow G(x) = e^{-x}$

$\Psi(x) = \int \frac{1}{3}(1-2x)e^{-x}$

$f(x) = 1-2x \qquad f'(x) = -2$

$g'(x) = e^{-x} \qquad g(x) = -e^{-x}$

$= \frac{1}{3}(-e^{-x}(1-2x) - \int 2e^{-x})$ $\qquad z = \frac{1}{3}(2x+1) + Ce^{x}$

$= \frac{1}{3}(-e^{-x}(1-2x) + 2e^{-x})$

$= \frac{1}{3}e^{-x}(2x-1+2) = \frac{1}{3}e^{-x}(2x+1) + C$

<u>Stap 5</u>: $z = -\frac{1}{3}y^{-3} = -\frac{1}{3y^3}$

$\Leftrightarrow y^3 = -\frac{1}{3z}$

$\Leftrightarrow y = \sqrt[3]{-\frac{1}{3(\frac{1}{3}(2x+1) + C.e^{x})}} = \frac{1}{\sqrt{-2x-1+Ce^{x}}}$

of $y = 0$

(2) $y' + y = y^4 e^{2x}$

Stap 1: $y^{-4} y' + y^{-3} = e^{2x}$

Stap 2: $y^4 = 0$ een oplossing v.d. DV?

$\Leftrightarrow 0' + 0 = 0^4 . e^{2x}$

$\Leftrightarrow 0 = 0 \rightarrow$ ja

Stap 3: $z = \frac{1}{1-p} y^{1-p} = \frac{1}{1-4} y^{1-4} = -\frac{1}{3} y^{-3}$

$\rightarrow z' = y^{-p} y' = y^{-4} y'$

$\rightarrow z' + (1-p).R(x).z = S(x)$

$\Leftrightarrow z' + (1-4).z = e^{2x}$

$\Leftrightarrow z' - 3z = e^{2x}$

Stap 4: $\Phi(x) = \int P(x)\,dx = -3x$

$\rightarrow G(x) = e^{-3x}$

$\Psi(x) = \int e^{-3x} e^{2x} dx = \int e^{-x} dx = -e^{-x} + C$

$z = \frac{1}{e^{-3x}} \left[-e^{-x} + C \right] = -e^{2x} + Ce^{3x}$

Stap 5: $z = -\frac{1}{3} y^{-3} = -\frac{1}{3y^3}$

$\Leftrightarrow y^3 = -\frac{1}{3z}$

$\Leftrightarrow y = \sqrt[3]{\frac{-1}{-3e^{2x} + Ce^{3x}}}$

$= \frac{1}{\sqrt[3]{3e^{2x} - Ce^{3x}}}$

$= \frac{1}{e^{x} \sqrt[3]{3e^{-x} + C}}$ of $y = 0$

③ $xy' - x^2y^2 = -y$

$\Leftrightarrow y' + \frac{1}{x}y = xy^2$

Stap 1 : $y^{-2}y' + \frac{1}{x}y^{-1} = x$

Stap 2 : $y^2 = 0$ een oplossing vd DV?

$x\,0' - x^2\,0^2 = 0$

$0 = 0 \rightarrow$ ja

Stap 3 : $z = \frac{1}{1-p}y^{1-p} = \frac{1}{1-2}y^{1-2} = -y^{-1}$

$\rightarrow z' = y^{-p}y' = y^{-2}y'$

$\rightarrow z' + (1-p)\,R(x)\,z = S(x)$

$\Leftrightarrow z' + (1-2).\frac{1}{x}\,z = x$

$\Leftrightarrow z' - \frac{1}{x}.z = x$

Stap 4 : $\Phi = \int -\frac{1}{x}dx = -\ln|x|$

$\rightarrow G(x) = e^{-\ln x} = \frac{1}{x}$

$\Psi(x) = \int \frac{1}{x}.x\,dx = x + C$

$z = x.(x+C) = x^2 + Cx$

Stap 5 : $z = -\frac{1}{y}$

$\Leftrightarrow y = -\frac{1}{z}$

$\Leftrightarrow y = -\frac{1}{x^2+Cx}$ of $y = 0$

④ $y' + y + y^2(\sin x - \cos x) = 0$

$\Leftrightarrow y' + y = -(\sin x - \cos x)y^2$

Stap 1: $y^{-2}y' + y^{-1} = \cos x - \sin x$

Stap 2: $y^2 = 0$ een oplossing vd DV?

$0' + 0 + 0^2(\sin x - \cos x) = 0$

$0 = 0 \rightarrow$ ja

Stap 3: $z = \frac{1}{1-p} y^{1-p} = -y^{-1}$

$\rightarrow z' = y^{-p} y' = y^{-2} y'$

$\rightarrow z' + (1-p)\, R(x)\, z = S(x)$

$\Leftrightarrow z' + (1-2).\, z = \cos x - \sin x$

$\Leftrightarrow z' - z = \cos x - \sin x$

Stap 4: $\Phi(x) = \int -dx = -x$

$\rightarrow G(x) = e^{-x}$

$\Psi(x) = \int e^{-x}(\cos x - \sin x)dx$

$f(x) = e^{-x} \qquad f'(x) = -e^{-x}$

$g'(x) = \cos x - \sin x \qquad g(x) = \sin x + \cos x$

$= e^{-x}(\sin x + \cos x) + \int e^{-x}(\sin x + \cos x)dx$

$f(x) = e^{-x} \qquad f'(x) = -e^{-x}$

$g'(x) = \sin x + \cos x \qquad g(x) = -\cos x + \sin x$

$= e^{-x}(\sin x + \cos x) + e^{x}(\sin x - \cos x) + \int e^{x}(\cos x - \sin x)$

$= e^{-x}(2 \sin x) + C$

$z = 2 \sin x + Ce^{x}$

Stap 5: $z = -\frac{1}{y}$

$\Leftrightarrow y = -\frac{1}{z} \Leftrightarrow y = -\frac{1}{2\sin x + Ce^{x}}$

⑤ $x\,dy - (y+y^2)\,dx = 0$

$\Leftrightarrow x\,dy = (y+y^2)\,dx$

$\Leftrightarrow \frac{dy}{dx} = \frac{1}{x} y + \frac{1}{x} y^2$

$\Leftrightarrow \frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x} y^2$

<u>Stap 1</u>: $y^{-2} y' - \frac{1}{x} y^{-1} = \frac{1}{x}$

<u>Stap 2</u>: $y^2 = 0$ een oplossing vd DV?

$0' - \frac{1}{x} 0 = \frac{1}{x} 0^2$

$0 = 0 \rightarrow$ ja

<u>Stap 3</u>: $z = \frac{1}{1-p} y^{1-p} = -y^{-1}$

$\rightarrow z' = y^{-p} y' = y^{-2} y'$

$\rightarrow z' + (1-p)\, R(x) z = S(x)$

$\Leftrightarrow z' - \left(-\frac{1}{x}\right) z = \frac{1}{x}$

$\Leftrightarrow z' + \frac{1}{x} z = \frac{1}{x}$

<u>Stap 4</u>: $\Phi(x) = \int \frac{1}{x}\,dx = \ln x$

$\rightarrow G(x) = e^{\ln x} = x$

$\Psi(x) = \int x \cdot \frac{1}{x}\,dx = x + C$

$z = \frac{1}{x}(x + C)$

<u>Stap 5</u>: $z = -\frac{1}{y}$

$\Leftrightarrow y = -\frac{1}{z}$

$\Leftrightarrow y = -\frac{1}{\frac{1}{x}(x+C)}$

$\Leftrightarrow y = -\frac{x}{x+C}$

⑥ $dy = (x^2y^4 - \frac{y}{x})dx$

$\Leftrightarrow \frac{dy}{dx} = x^2y^4 - \frac{y}{x}$

$\Leftrightarrow y' + \frac{1}{x}y = x^2y^4$

Stap 1: $y^{-4}y' + \frac{1}{x}y^{-3} = x^2$

Stap 2: $y^4 = 0$ een oplossing vd DV?

$0' + \frac{1}{x}0 = x^2 \cdot 0^4$

$0 = 0 \rightarrow$ ja

Stap 3: $z = \frac{1}{1-p}y^{1-p} = -\frac{1}{3}y^{-3}$

$z' = y^{-p}y' = y^{-4}y'$

$\rightarrow z' + (1-p)\,R(x)\,z = S(x)$

$\Leftrightarrow z' - 3 \cdot \frac{1}{x} z = x^2$

Stap 4: $\Phi(x) = \int -\frac{3}{x} = -3\ln x$

$\rightarrow G(x) = \frac{1}{x^3}$

$\Psi(x) = \int \frac{1}{x^3} \cdot x^2 dx = \int \frac{1}{x}dx = \ln x + C$

$z = x^3 \cdot \ln x + C x^3$

Stap 5: $z = -\frac{1}{3}y^{-3}$

$\Leftrightarrow y^3 = -\frac{1}{3z}$

$\Leftrightarrow y^3 = -\frac{1}{3(x^3 \ln x + x^3 C)}$

$= -\frac{1}{x\sqrt{\ln x \cdot 3 + C}}$

$= \frac{1}{x\sqrt{-3\ln|x| + C}}$

(7) $\frac{dy}{dx} + 2xy + xy^4 = 0$

$\Leftrightarrow y' + 2xy = -xy^4$

<u>Stap 1</u>: $y^{-4}y' + 2xy^{-3} = -x$

<u>Stap 2</u>: $y^4 = 0$ oplossing vd DV?

$0' + 2x.0 = -x.0^4$

$0 = 0 \rightarrow$ ja

<u>Stap 3</u>: $z = \frac{1}{1-p} y^{1-p} = -\frac{1}{3} y^{-3}$

$\rightarrow z' = y^{-p}y' = y^{-4}y'$

$\rightarrow z' + (1-p)R(x)z = S(x)$

$\Leftrightarrow z' - 6xz = -x$

<u>Stap 4</u>: $\Phi(x) = \int -6x\,dx = -3x^2$

$\rightarrow G(x) = e^{-3x^2}$

$\Psi(x) = \int e^{-3x^2}(-x)\,dx$ * stel $u = -3x^2$, $du = -6x\,dx$

$= \frac{1}{6}\int e^u du$

$= \frac{1}{6} e^{-3x^2} + C$

$z = \frac{1}{e^{-3x^2}}\left(\frac{1}{6}e^{-3x^2} + C\right) = \frac{1}{6} + C.e^{3x^2}$

<u>Stap 5</u>: $z = -\frac{1}{3}y^{-3}$

$\Leftrightarrow y^3 = -\frac{1}{3z}$

$\Leftrightarrow y^3 = -\frac{1}{3\left(\frac{1}{6} + C.e^{3x^2}\right)}$

$= \frac{1}{-\frac{1}{2} + C.e^{3x^2}}$

$\Leftrightarrow y = \sqrt[3]{\frac{1}{-1/2 + C.e^{3x^2}}}$

(8) $-2xy' = y + y^3x^3\sin^2x.\cos x$

$\Leftrightarrow y' + \frac{1}{2x}y = -\frac{1}{2x}.x^3y^3\sin^2x.\cos x$

<u>Stap 1</u>: $y^{-3}y + \frac{1}{2x}y^{-2} = -\frac{1}{2}x^2\sin^2x\cos x$

<u>Stap 2</u>: $y^3=0$ oplossing vd DV?

$0' + \frac{1}{2x}0 = 0 \rightarrow$ ja

<u>Stap 3</u>: $z = \frac{1}{1-p}y^{1-p} = -\frac{1}{2}y^{-2}$

$\rightarrow z' = y^{-3}y'$

$\rightarrow z' + (1-3).\frac{1}{2x}z = -\frac{1}{2}x^2\sin^2x.\cos x$

$\Leftrightarrow z' - \frac{1}{x}z = -\frac{1}{2}x^2\sin^2x\cos x$

<u>Stap 4</u>: $\Phi(x) = \int P(x)dx = -\ln x$

$\rightarrow G(x) = \pm\frac{1}{x}$

$\Psi(x) = \int \pm\frac{1}{x}(-\frac{1}{2})x^2\sin^2x\cos x$

$= \mp\frac{1}{2}\int x\sin^2x\cos x$

$f(x) = x \quad f'(x) = 1$

$g'(x) = \sin^2x\cos x \quad g(x) = \frac{\sin^3x}{3}$

$\rightarrow \int u^2du$

$= \mp\frac{1}{2}\left(x.\frac{\sin^3x}{3} - \int\frac{\sin^3x}{3}dx\right)$

$= \mp\frac{1}{6}x.\sin^3x \pm \frac{1}{6}\underbrace{\int\sin^3x\,dx}_{I}$

$I = \int\sin^3x\,dx = \int\sin x(1-\cos^2x)\,dx$

$= \int\sin x\,dx - \int\cos^2x\sin x\,dx = -\cos x + \frac{\cos^3x}{3}$

$\Psi(x) = \mp\frac{1}{6}x\sin^3x \pm \frac{1}{6}(-\cos x + \frac{\cos^3x}{3})$

$= \mp\frac{1}{6}x.\sin^3x \mp \frac{1}{6}\cos x \pm \frac{1}{18}\cos^3x$

$x = \pm x\left(\mp\frac{1}{6}x\cdot\sin^3 x \mp \frac{1}{6}\cos x \pm \frac{1}{18}\cos^3 x + C\right)$

Stap 5: FOERT!

(9) $yy' - xy^2 + x = 0$

$\Leftrightarrow y' - xy = -xy^{-1}$

Stap 1: $yy' - xy^2 = -x$

Stap 2: $\frac{1}{y} = 0$ een oplossing van de DV?

$\rightarrow$ nee

Stap 3: $z = \frac{1}{1-p}y^{1-p} = \frac{1}{2}y^2$

$z' = y^{+1}y'$

$z' + (1-p)R(x)z = S(x)$

$\Leftrightarrow z' + 2.(-x)z = -x$

$\Leftrightarrow z' - 2x.z = -x$

Stap 4: $\Phi(x) = \int -2x\,dx = -x^2$

$\rightarrow G(x) = e^{-x^2}$

$\Psi(x) = \int (-x)e^{-x^2}dx$ stel $u = -x^2$, $du = -2x\,dx$

$= \frac{1}{2}\int e^u du$

$= \frac{1}{2}e^{-x^2} + C$

$z = \frac{1}{e^{-x^2}}\left(\frac{1}{2}e^{-x^2} + C\right)$

$= \frac{1}{2} + C.e^{x^2}$

Stap 5: $z = \frac{1}{2}y^2$

$\Leftrightarrow y^2 = 2z$

$\Leftrightarrow y = \pm\sqrt{2z}$

$\Leftrightarrow y = \pm\sqrt{1 + C.e^{x^2}}$

(10) $-3y' - yx(2+e^{x^2}y^3)=0$

$\Leftrightarrow -3y' = 2xy + x e^{x^2} y^4$

$\Leftrightarrow y' + \frac{2}{3}xy = -\frac{1}{3}x e^{x^2} y^4$

Stap 1: $y^{-4}y' + \frac{2}{3}xy^{-3} = -\frac{1}{3}xe^{x^2}$

Stap 2: $y^4=0$ een oplossing vd DV?

$0'+0=0$

$\to$ ja

Stap 3: $z = \frac{1}{1-p} y^{1-p} = -\frac{1}{3}y^{-3}$

$z' = y^{-4}y'$

$z' + (1-4)\frac{2}{3}x z = -\frac{1}{3}x e^{x^2}$

$\Leftrightarrow z' - 2xz = -\frac{1}{3} x e^{x^2}$

Stap 4: $\Phi(x) = \int P(x)dx = -x^2$

$\to G(x) = e^{-x^2}$

$\Psi(x) = \int e^{-x^2}\left(-\frac{1}{3}\right) x e^{x^2} dx$

$= \int -\frac{1}{3} x \, dx$

$= -\frac{1}{6} x^2 + C$

$z = \frac{1}{e^{-x^2}}\cdot\left(-\frac{1}{6} x^2 + C\right)$

$= e^{x^2}\left(-\frac{1}{6} x^2+C\right)$

Stap 5: $z = \frac{1}{-3y^3}$

$\Leftrightarrow y^3 = -\frac{1}{3z}$

$\Leftrightarrow y^3 = \frac{1}{(\frac{x^2}{2}+C)e^{x^2}}$

$\Leftrightarrow y = \frac{1}{\sqrt[3]{(\frac{x^2}{2}+C)e^{x^2}}}$

3.4 /

3.5 Gegeven: p = eenheidsprijs

$p(0) = p$. aanbod A_0

vraag: $q = a - bp$ $\Big\}$ $a, b, A \in \mathbb{R}$

Gevraagd: $p(t)$ en limietprijs?

Oplossing: $\frac{dp}{dt} = p' = k.(a - bp - A_0)$

$\Leftrightarrow p' = ka - kbp - kA_0$

$\Leftrightarrow p' + kbp = k(a - A_0)$

Stap 1: $\Phi(t) = \int kb\,dt = kb\int dt$

$= kbt$

$\rightarrow G(t) = e^{kbt}$

Stap 2: $\Psi(t) = \int k.(a - A_0).e^{kbt}\,dt$

$= k.(a - A_0)\int e^{kbt}dt$ stel $u = bkt$, $du = kbdt$

$= \frac{(a - A_0)}{b} e^{kbt} + C$

Stap 3: $p = \frac{1}{G(t)}[\Psi(t) + C]$

$= e^{-kbt}\left[\frac{(a - A_0)}{b}e^{kbt} + C\right]$

RVW: $t = 0 \rightarrow p = p_0$

$p_0 = \frac{a - A_0}{b} + C\,\underbrace{e^{-bk0}}_{1} \rightarrow C = p_0 - \frac{a - A_0}{b}$

$\rightarrow p(t) = \frac{a - A_0}{b} + \left(p_0 + \frac{a - A_0}{b}\right) e^{-bkt}$

(3.6) Gegeven: $P(t)$: aantal personen met een zeker kenmerk op tydstip t

$P=N$

$$\frac{dP}{dt} = x(P,t).P$$
$$P(0) = P_0$$

(1) $\frac{dP}{dt} = x.P$

$\Leftrightarrow \int \frac{dP}{P} = \int x dt$

$\Leftrightarrow \ln|P| = xt + C \quad (P>0 \,;\, C \in \mathbb{R})$

$\Leftrightarrow P = e^{xt+C}$

$= e^{C}.e^{xt}$

$= C.e^{xt} \quad (C \in \mathbb{R}_0^+)$

Uit de randvoorwaarde $P(0) = P_0$ vinden we $C_1 = P_0$, dus:
$P(t) = P_0 . e^{xt}$

(2) $x = aP + b \quad (a, b \in \mathbb{R}_0^+)$

$\frac{dP}{dt} = (aP+b).P$

$\Leftrightarrow P' - bP = aP^2$

$\Leftrightarrow P^{-2}P' - bP^{-1} = a$

Stap 1: $x = \frac{1}{1-p} y^{1-p} = -\frac{1}{P}$

$x' = P^{-2}P'$

$x' + (1-p).(-b)x = a \Leftrightarrow x' + bx = a$

Stap 2: $\Phi(t) = \int b dt = bt$

$\rightarrow G(t) = e^{bt}$

$\Psi(t) = \int e^{bt} a dt = \frac{a}{b}e^{bt} + C$

$x = \frac{\frac{a}{b}e^{bt} + C}{e^{bt}} = \frac{a}{b} + Ce^{-bt}$

<u>Stap 3</u>: $P(t) = -\frac{1}{z}$

$$\Leftrightarrow P(t) = \frac{-1}{\frac{a}{b} + Ce^{-bt}} = \frac{-b}{a + Cbe^{-bt}}$$

$$P(t) = \frac{-b\,e^{bt}}{ae^{bt} + Cb}$$

Invoeren van de RVW zal de constante bepalen:

$t = 0 \rightarrow P = P_0$

$P_0 = \frac{Cb}{1 - Ca}$ $\Leftrightarrow (1 - Ca)P_0 = Cb$

$\Leftrightarrow P_0 = Cb + CaP_0$

$\Leftrightarrow C = \frac{P_0}{aP_0 + b}$

$$\rightarrow P(t) = \frac{P_0\,b e^{bt}}{aP_0 + b - P_0 a e^{bt}}$$

37 $k(t)$: kapitaalstock

$K(0) = K_0$

<u>Oplossing</u>: $\frac{dK}{dt} = b - aK$

$\int \frac{dK}{b - aK} = \int dt$ stel $u = b - aK$, $du = -a\,dK$

$\Leftrightarrow -\frac{1}{a}\int \frac{du}{u} = t$ Is $b - aK = 0$ een oplossing? $K = \frac{b}{a} \rightarrow$ ja

$\Leftrightarrow -\frac{1}{a}\ln|b - aK| = t + C$

$\Leftrightarrow |b - aK| = e^{-at}e^{C}$

$\Leftrightarrow b - aK = \pm C.e^{-at}$

$\Leftrightarrow -aK = -b + Ce^{-at}$

$\Leftrightarrow K = \frac{b}{a} + \frac{C}{a}e^{-at}$

<u>Samengevat</u>: $K(t) = \frac{b}{a} + Ce^{-at}$

<u>RVW</u>: $t = 0 \rightarrow K = K_0$

$K_0 = \frac{b}{a} + Ce^{0}$

$\rightarrow C = K_0 - \frac{b}{a}$ $\rightarrow K(t) = \frac{b}{a} + \left(K_0 - \frac{b}{a}\right)e^{-at}$

3.8 /

3.9 $\frac{dN}{dt} = a(20-N)$

$\Leftrightarrow \int \frac{dN}{20-N} = a\int dt$ stel $u = 20-N$, $du = -dN$

$\Leftrightarrow -\ln|20-N| = at$

(3.10) Aantal universiteitsstudenten : 50 000, dus 5

→ $(5-N)$: aantal studenten die de telfoon nog niet gebruiken

$$\frac{N'}{N} = k(5-N) = 5k\left(1-\frac{N}{5}\right) = a.\left(1-\frac{N}{5}\right)$$

$$\Leftrightarrow N' = aN - a\frac{N^2}{5}$$

Stap 0: $N' - aN = -a\frac{N^2}{5}$

Stap 1: $N^{-2}N' - aN^{-1} = -\frac{a}{5}$ mits $N^2 \neq 0$

Stap 2 : logistische groei onderstelt $N \neq 0$

Stap 3: $x = \frac{1}{1-\rho} y^{1-\rho} = \frac{N^{-1}}{-1}$

$$x' = N^{-2}N'$$

$$x' + (-1)(-a)x = -\frac{a}{5} \Leftrightarrow x' + ax = -\frac{a}{5}$$

Stap 4: $\Phi(x) = \int a\,dt = at$

$$\rightarrow G(x) = e^{at}$$

$$\Psi(x) = \int e^{at}\left(-\frac{a}{5}\right)dt = -\frac{a}{5a}e^{at} = -\frac{1}{5}e^{at} + C$$

$$x = \frac{1}{e^{at}}\left[-\frac{1}{5}e^{at} + C\right]$$

$$= -\frac{1}{5} + Ce^{-at}$$

Stap 5: $x = -\frac{1}{N}$

$$\Leftrightarrow N = -\frac{1}{x}$$

$$\Leftrightarrow N = \frac{-1}{-\frac{1}{5} + Ce^{-at}}$$

$$= \frac{5}{1 + Ce^{-at}}$$

RVW:

$$N(0) = \frac{500}{10000} = 0{,}05 \rightarrow \frac{5}{100} = \frac{5}{1+Ce^0} = \frac{5}{1+C} \rightarrow C = 99$$

$$\rightarrow N(t) = \frac{5}{1+99e^{-at}}$$

$$N(1) = \frac{1500}{10000} = 0{,}15 \rightarrow \frac{15}{100} = \frac{5}{1+99e^{-a1}}$$

$$\rightarrow 1+99e^{-a} = \frac{100}{3}$$

$$\rightarrow 99e^{-a} = \frac{100}{3} - 1 = \frac{97}{3}$$

$$\rightarrow e^{-a} = \frac{97}{3.99} = \frac{97}{297}$$

<u>Oplossing:</u>

$$N(t) = \frac{5}{1+99\left(\frac{97}{297}\right)^t}$$

(3.11) y = # studenten op de hoogte v/h nieuws na t dagen

$$\frac{y'}{y} = k(4000-y)$$

$$= \underbrace{4000k}_{=a}\left(1-\frac{y}{4000}\right)$$

$$= a.\left(1-\frac{y}{4000}\right)$$

$$y' = ay - \frac{ay^2}{4000}$$

$$y' - ay = \frac{-a}{4000}y^2$$

<u>Stap 1</u>: $y^{-2}y' - ay^{-1} = \frac{-a}{4000}$

<u>Stap 2</u>: $y^2 = 0$ een oplossing vd DV?

$0' - a0 = 0 \rightarrow$ ja $\rightarrow$ hier niet v. toepassing door RVW

<u>Stap 3</u>: $x = \frac{y^{-1}}{-1}$

$$x' = y^{-2}y'$$

$$x' + ax = \frac{-a}{4000}$$

<u>Stap 4</u>: $\Phi(x) = \int a\,dt = at$

$$\rightarrow G(x) = e^{at}$$

$$\Psi(x) = \frac{-a}{4000}\int e^{at}dt \qquad \text{stel } u = at,\ du = a\,dt$$

$$= \frac{-a}{4000}\,\frac{1}{a}\int e^u du$$

$$= \frac{-e^u}{4000} = -\frac{1}{4000}e^{at} + C$$

$$x = \frac{1}{e^{at}}\left[-\frac{1}{4000}e^{at} + C\right]$$

$$= -\frac{1}{4000} + C.e^{-at}$$

<u>Stap 5:</u> $x = -\frac{1}{y}$

$$\Leftrightarrow y = -\frac{1}{x}$$

$$= \frac{-4000}{-1+Ce^{-at}} = \frac{4000}{1+Ce^{-at}}$$

$y(t) = \frac{4000}{1+Ce^{-at}}$ met $t = 0 \rightarrow y = 2$

$$\Leftrightarrow 2 = \frac{4000}{1+C.e^0}$$

$$\Leftrightarrow 1+C = 2000$$

$$\Leftrightarrow C = 1999$$

$$\rightarrow y(t) = \frac{4000}{1+1999e^{-at}}$$

RVW als $t = 1 \rightarrow y = 20$

$$\Leftrightarrow 20 = \frac{4000}{1+1999.e^{-a}}$$

$$\Leftrightarrow 1+1999.e^{-a} = 200$$

$$\Leftrightarrow 1.999e^{-a} = 199$$

$$\Leftrightarrow e^{-a} = \frac{199}{1999}$$

$$\Leftrightarrow -a = \ln\left(\frac{199}{1999}\right)$$

$$\rightarrow y(t) = \frac{4000}{1+1999e^{t\ln\left(\frac{199}{1999}\right)}}$$

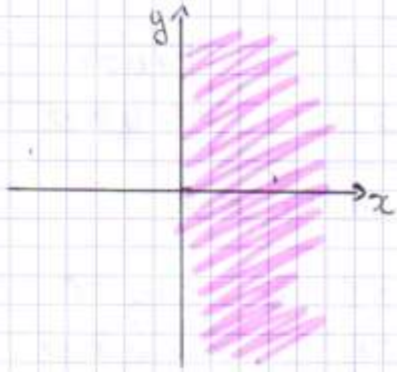
$$= \frac{4000}{1+1999\left(\frac{199}{1999}\right)^t}$$

Hoofdstuk 4: Functies van twee veranderlijken

(4.1) ① $z = xy + 2\ln x$

BVW: $x > 0$

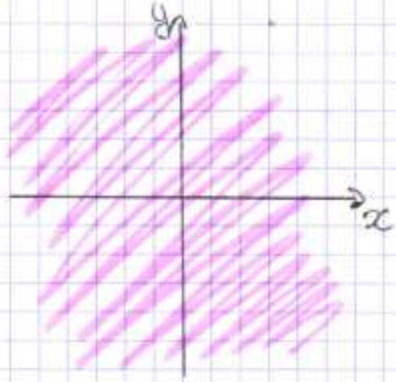
$\Rightarrow \text{dom} f = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$



② $z = y \sin(3x - 2y)$

BVW: /

$\Rightarrow \text{dom} f = \mathbb{R}^2$



③ $z = x\, e^{xy}$

BVW: /

$\Rightarrow \text{dom} f = \mathbb{R}^2$

(zie grafiek vorige oefening)

④ $z = (x^2-y^2)^{3/2}$

$= \sqrt{(x^2-y^2)^3}$

<u>BVW</u>: $(x^2-y^2)^3 \geq 0$

$\Leftrightarrow x^2-y^2 \geq 0$

$\Leftrightarrow (x-y)(x+y) \geq 0$

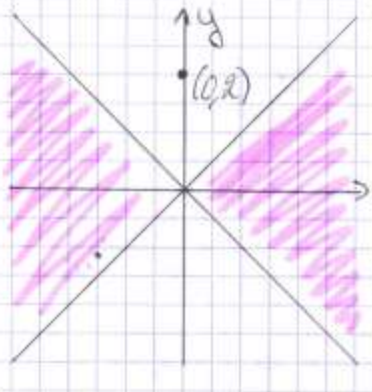
- <u>1e geval</u>: $x-y \geq 0$ en $x+y \geq 0$

$$\underbrace{\Leftrightarrow x \geq y \qquad \Leftrightarrow y \geq -x}_{-x \leq y \leq x}$$

- <u>2e geval</u>: $x-y \leq 0$ en $x+y \leq 0$

$$\underbrace{\Leftrightarrow x \leq y \qquad \Leftrightarrow y \leq -x}_{x \leq y \leq -x}$$

$\text{dom} f = \{(x,y) \in \mathbb{R}^2 \mid -x \leq y \leq x\} \cup \{(x,y) \in \mathbb{R}^2 \mid x \leq y \leq -x\}$



⑤ $z = \ln\left(\frac{y-x}{y+x}\right)$

<u>BVW</u>: $\frac{y-x}{y+x} > 0$

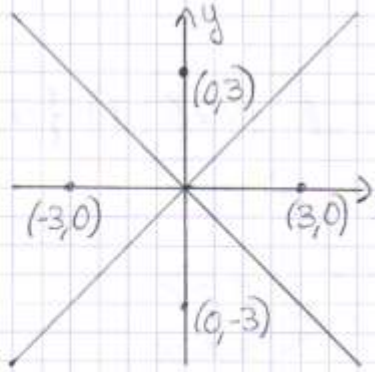
- <u>1e geval</u>: $y-x > 0$ en $y+x > 0$

$$\underbrace{\Leftrightarrow y > x \qquad \Leftrightarrow y > -x}_{-y < x < y} \Leftrightarrow x > -y$$

- <u>2^e geval:</u> $y-x<0$ en $y+x<0$

$$\underbrace{\Leftrightarrow y<x \qquad \Leftrightarrow x<-y}_{y<x<-y}$$

$\text{dom} f = \{(x,y)\in\mathbb{R}^2 \mid -y<x<y\} \cup \{(x,y)\in\mathbb{R}^2 \mid y<x<-y\}$



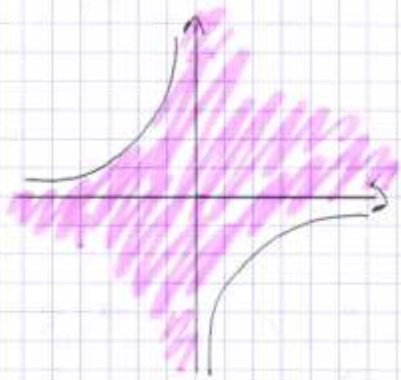
⑥ $z=\ln(xy+1)$

<u>BVW:</u> $xy+1>0$

$\Leftrightarrow xy>-1$

$\Leftrightarrow y>-\frac{1}{x}$

$\text{dom} f = \{(x,y)\in\mathbb{R}^2 \mid xy>-1\}$



⑦ $z = e^{-(x+y)^2}$

<u>BVW:</u> /

$\mathrm{dom} f = \mathbb{R}^2$

⑧ $z = \dfrac{x^3-y^3}{xy}$

<u>BVW:</u> $xy \neq 0$

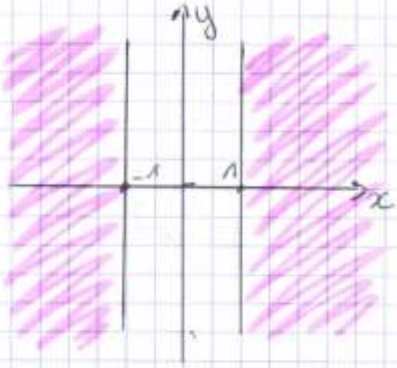
$\mathrm{dom} f = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\} \cap \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$

⑨ $z = \sqrt{x^2-1} - \sqrt[3]{1-y^2}$

<u>BVW:</u> $x^2 - 1 \geq 0$

$\Leftrightarrow x^2 \geq 1 \left\langle \begin{array}{l} x \geq 1 \\ x \leq -1 \end{array} \right.$

$\mathrm{dom} f = \{(x,y) \in \mathbb{R}^2 \mid x \leq -1\} \cup \{(x,y) \in \mathbb{R}^2 \mid x \geq 1\}$



⑩ $z = \sqrt{\dfrac{1}{x} + \dfrac{1}{y}}$

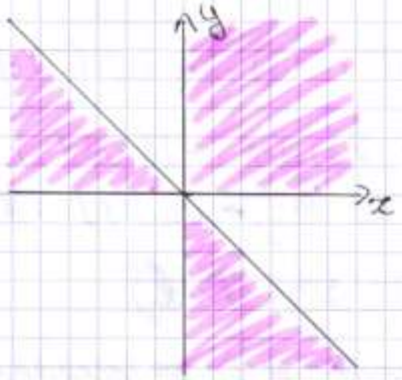
<u>BVW:</u> * $x \neq 0$

* $y \neq 0$
* $\dfrac{1}{x} + \dfrac{1}{y} \geq 0 \Leftrightarrow \dfrac{x+y}{xy} \geq 0$

- 1e geval: $y+x \geq 0$ en $xy > 0$
  - (1a) $x>0$; $y>0$ en $y+x \geq 0$ → altijd voldaan
  - (1b) $x<0$; $y<0$ en $y+x \geq 0$ → kan niet

- 2e geval: $y+x \leq 0$ en $xy < 0$
  - (2a) $x>0$; $y<0$ en $y+x \leq 0$ → mogelijk
  - (2b) $x<0$; $y>0$ en $y+x \leq 0$ → mogelijk

$$\text{dom} f = \{(x,y) \in \mathbb{R}^2 \mid x>0, y>0\} \cup \{(x,y) \in \mathbb{R}^2 \mid x>0, y<0, y+x \leq 0\} \cup \{(x,y) \in \mathbb{R}^2 \mid x<0, y>0, y+x \leq 0\}$$



(4.2) (1) $z = xy + 2\ln x$

$$\frac{dz}{dx} = y + \frac{2}{x}$$

$$\frac{dz}{dy} = x$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(y + \frac{2}{x}\right) = -\frac{2}{x^2}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}(x) = 0$$

$$\frac{d^2z}{dx\,dy} = \frac{d}{dx}(x) = 1$$

② $z = y \sin(3x-2y)$

$$\frac{dz}{dx} = y \cos(3x-2y)\, 3$$
$$= 3y \cdot \cos(3x-2y)$$

$$\frac{dz}{dy} = \sin(3x-2y) + y \cdot \cos(3x-2y)\,(-2)$$
$$= \sin(3x-2y) - 2y \cdot \cos(3x-2y)$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(3y \cos(3x-2y)\right)$$
$$= 3y \cdot (-\sin(3x-2y)) \cdot 3$$
$$= -9y \cdot \sin(3x-2y)$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(\sin(3x-2y) - 2y \cdot \cos(3x-2y)\right)$$
$$= \cos(3x-2y) \cdot (-2) - \left(2 \cos(3x-2y) + 2y \cdot (-\sin(3x-2y))(-2)\right)$$
$$= -4 \cos(3x-2y) - 4y \sin(3x-2y)$$

$$\frac{d^2z}{dx\,dy} = \cancel{\frac{d}{dy}\left(-9y \sin(3x-2y)\right)}$$
$$= \cancel{-9 \sin(3x-2y) - 9y \cos(3x-2y)(-2)}$$
$$= \frac{d}{dy}\left(3y \cos(3x-2y)\right)$$
$$= 3 \cdot \cos(3x-2y) + 3y \cdot (-\sin(3x-2y)) \cdot (-2)$$
$$= 3 \cdot \cos(3x-2y) + 6y \cdot \sin(3x-2y)$$

③ $z = x \cdot e^{xy}$

$$\frac{dz}{dx} = e^{xy} + x \cdot y \cdot e^{xy}$$
$$= e^{xy}(1 + xy)$$

$$\frac{dz}{dy} = x \cdot x \cdot e^{xy}$$
$$= x^2 e^{xy}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(e^{xy} + xy \cdot e^{xy}\right)$$
$$= y \cdot e^{xy} + y \cdot e^{xy} + xy\, y\, e^{xy} = y \cdot e^{xy}(2 + xy)$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(x^2 e^{xy}\right)$$
$$= x^3 e^{xy}$$
$$\frac{d^2z}{dxdy} = \frac{d}{dx}\left(x^2 e^{xy}\right)$$
$$= 2x\, e^{xy} + x^2 y\, e^{xy}$$
$$= x\, e^{xy}(2 + xy)$$

(4) $z = (x^2 - y^2)^{3/2}$

$$\frac{dz}{dx} = \frac{3}{2}(x^2-y^2)^{1/2} \cdot 2x$$
$$= 3x\sqrt{x^2-y^2}$$
$$\frac{dz}{dy} = \frac{3}{2}(x^2-y^2)^{1/2}(-2y)$$
$$= -3y\sqrt{x^2-y^2}$$
$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(3x\sqrt{x^2-y^2}\right)$$
$$= 3\sqrt{x^2-y^2} + 3x \cdot \frac{1}{2}(x^2-y^2)^{-1/2} \cdot 2x$$
$$= 3\sqrt{x^2-y^2} + \frac{3x^2}{\sqrt{x^2-y^2}}$$
$$= \frac{3(x^2-y^2)+3x^2}{\sqrt{x^2+y^2}}$$
$$= \frac{6x^2-3y^2}{\sqrt{x^2+y^2}}$$
$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(-3y\sqrt{x^2-y^2}\right)$$
$$= -3\sqrt{x^2-y^2} + (-3)y \cdot \frac{1}{2}(x^2-y^2)^{-1/2}(-2y)$$
$$= -3\sqrt{x^2-y^2} + \frac{3y^2}{\sqrt{x^2-y^2}}$$
$$= \frac{-3(x^2-y^2)+3y^2}{\sqrt{x^2-y^2}}$$
$$= \frac{6y^2-3x^2}{\sqrt{x^2-y^2}}$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(3x\sqrt{x^2-y^2}\right)$$

$$= 3x \cdot \frac{1}{2} \cdot (x^2-y^2)^{-1/2} \cdot (-2y)$$

$$= \frac{-3xy}{\sqrt{x^2-y^2}}$$

⑤ $z = \ln \frac{y-x}{y+x} = \ln|y-x| - \ln|y+x|$

$$\frac{dz}{dx} = \frac{y+x}{y-x} \cdot \frac{(-1)(y+x)-(y-x) \cdot 1}{(y+x)^2}$$

$$= \frac{-y-x-y+x}{(y-x)(y+x)}$$

$$= \frac{-2y}{y^2-x^2}$$

$$\frac{dz}{dy} = \frac{y+x}{y-x} \cdot \frac{(1)(y+x)-(y-x) \cdot 1}{(y+x^2)^2}$$

$$= \frac{y+x-y+x}{y^2-x^2}$$

$$= \frac{2x}{y^2-x^2}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{2y}{x^2-y^2}\right)$$

$$= \frac{d}{dx}\left(2y \cdot (x^2-y^2)^{-1}\right)$$

$$= 2y \cdot (-1) \cdot (x^2-y^2)^{-2} \cdot 2x$$

$$= \frac{-4xy}{(x^2-y^2)^2}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(\frac{2x}{y^2-x^2}\right)$$

$$= \frac{d}{dy}\left(2x \cdot (y^2-x^2)^{-1}\right)$$

$$= 2x \cdot (-1) \cdot (y^2-x^2)^{-2} \cdot 2y$$

$$= \frac{-4xy}{(y^2-x^2)^2}$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(\frac{2y}{x^2-y^2}\right)$$

$$= \frac{2(x^2-y^2) - 2y(-2y)}{(x^2-y^2)^2}$$

$$= \frac{2x^2-2y^2+4y^2}{(x^2-y^2)^2}$$

$$= \frac{2x^2+2y^2}{(x^2-y^2)^2}$$

⑥ $z = \ln(xy+1)$

$$\frac{dz}{dx} = \frac{1}{xy+1} \cdot y$$

$$= \frac{y}{xy+1}$$

$$\frac{dz}{dy} = \frac{1}{xy+1} \cdot x$$

$$= \frac{x}{xy+1}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{y}{xy+1}\right)$$

$$= \frac{0.(xy+1) - y.y}{(xy+1)^2} = \frac{-y^2}{(xy+1)^2}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(\frac{x}{xy+1}\right)$$

$$= \frac{0.(xy+1) - x.x}{(xy+1)^2} = \frac{-x^2}{(xy+1)^2}$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(\frac{y}{xy+1}\right)$$

$$= \frac{1.(xy+1) - y.x}{(xy+1)^2} = \frac{1}{(xy+1)^2}$$

⑦ $z = e^{-(x+y)^2}$

$$\frac{dz}{dx} = e^{-(x+y)^2}(-2(x+y))$$

$$\frac{dz}{dy} = e^{-(x+y)^2}\cdot(-2(x+y))$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(-2(x+y)\, e^{-(x+y)^2}\right.$$

$$= -2\, e^{-(x+y)^2} + (-2)(x+y)\, e^{-(x+y)^2}\cdot(-2)\cdot(x+y)$$

$$= 4\,(x+y)^2 e^{-(x+y)^2} - 2\, e^{-(x+y)^2}$$

$$\hookrightarrow = \frac{d^2z}{dy^2} = \frac{d^2z}{dx\,dy}$$

⑧ $z = \frac{x^3 - y^3}{xy} = (x^3-y^3)\,(xy)^{-1}$

$$\frac{dz}{dx} = 3x^2\,(xy)^{-1} + (x^3-y^3)\,(-1)\,(xy)^{-2}\,y$$

$$= \frac{3x^2}{xy} - \frac{(x^3-y^3)y}{x^2y^2}$$

$$= \frac{3x}{y} - \frac{(x^3-y^3)}{x^2y}$$

$$= \frac{3x^3 - x^3 + y^3}{x^2y} = \frac{2x^3+y^3}{x^2y}$$

$$\frac{dz}{dy} = (-3y^2)\,(xy)^{-1} + (x^3-y^3)\,(-1)\,(xy)^{-2}\,x$$

$$= -\frac{3y}{x} - \frac{(x^3-y^3)x}{x^2y^2}$$

$$= \frac{-3y^3 - x^3 + y^3}{xy^2}$$

$$= \frac{-2y^3 - x^3}{xy^2} = -\,\frac{2y^3 + x^3}{xy^2}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{2x^3+y^3}{x^2y}\right) = \frac{d}{dx}\left((2x^3+y^3)(x^2y)^{-1}\right)$$
$$= 6x^2(x^2y)^{-1} + (2x^3+y^3)\cdot(-1)\cdot(x^2y)^{-2}\cdot(2xy)$$
$$= \frac{6}{y} - \frac{2(2x^3+y^3)}{x^3y}$$
$$= \frac{6x^3-4x^3-2y^3}{x^3y} = \frac{2x^3-2y^3}{x^3y}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(\frac{-2y^3-x^3}{xy^2}\right) = \frac{d}{dy}\left(-(2y^3+x^3)(xy^2)^{-1}\right)$$
$$= -6y^2(xy^2)^{-1} + (-(2y^3+x^3))\cdot(-1)\cdot(xy^2)^{-2}\cdot(2xy)$$
$$= -\frac{6}{x} + \frac{2(2y^3+x^3)}{xy^3}$$
$$= -\frac{6y^3+4y^3+2x^3}{xy^3}$$
$$= \frac{2x^3-2y^3}{xy^3}$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(\frac{2x^3+y^3}{x^2y}\right) = \frac{d}{dy}\left((2x^3+y^3)(x^2y)^{-1}\right)$$
$$= 3y^2(x^2y)^{-1} + (2x^3+y^3)(-1)(x^2y)^{-2}x^2$$
$$= \frac{3y}{x^2} - \frac{(2x^3+y^3)}{x^2y^2}$$
$$= \frac{3y^3-y^3-2x^3}{x^2y^2}$$
$$= \frac{2y^3-2x^3}{x^2y^2}$$

9) $z = \sqrt{x^2-1} - \sqrt[3]{1-y^2}$

$$\frac{dz}{dx} = \frac{1}{2}(x^2-1)^{-1/2}\,2x$$
$$= \frac{x}{\sqrt{x^2-1}}$$

$$\frac{dz}{dy} = -\frac{1}{3}(1-y^2)^{-2/3}\cdot(-2y)$$
$$= \frac{2}{3}\frac{y}{(1-y^2)^{2/3}}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{d}{dx}\left(x.(x^2-1)^{-1/2}\right)$$
$$= (x^2-1)^{-1/2} + x\left(-\frac{1}{2}\right).(x^2-1)^{-3/2}(2x)$$
$$= \frac{1}{(x^2-1)^{1/2}} - \frac{x^2}{(x^2-1)^{3/2}}$$
$$= \frac{x^2-1-x^2}{(x^2-1)^{3/2}}$$
$$= \frac{-1}{(x^2-1)^{3/2}}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(\frac{2y}{3(1-y^2)^{2/3}}\right) = \frac{d}{dy}\left(\frac{2}{3}y\cdot(1-y^2)^{-2/3}\right)$$
$$= \frac{2}{3}(1-y^2)^{-2/3} + \frac{2}{3}y\left(-\frac{2}{3}\right)(1-y^2)^{-5/3}(-2y)$$
$$= \frac{2}{3(1-y^2)^{2/3}} + \frac{8y^2}{9(1-y^2)^{5/3}}$$
$$= \frac{6(1-y^2)+8y^2}{9(1-y^2)^{5/3}}$$
$$= \frac{6+2y^2}{9(1-y^2)^{5/3}} = \frac{2(y^2+3)}{9(1-y^2)^{5/3}}$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{d}{dy}\left(x.(x^2-1)^{-1/2}\right)$$
$$= 0$$

(10) $\cancel{\frac{dz}{dx}}$ $z = \sqrt{\frac{1}{x}+\frac{1}{y}} = \left(\frac{1}{x}+\frac{1}{y}\right)^{1/2}$

$$\frac{dz}{dx} = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1/2}.(-1)\,\frac{1}{x^2}$$
$$= -\frac{1}{2x^2\sqrt{\frac{1}{x}+\frac{1}{y}}}$$

$$\frac{dz}{dy} = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1/2}(-1).\frac{1}{y^2}$$
$$= -\frac{1}{2y^2\sqrt{\frac{1}{x}+\frac{1}{y}}}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(-\frac{1}{2}x^{-2}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1/2}\right)$$
$$= x^{-3}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1/2} - \frac{1}{2}x^{-2}\left(-\frac{1}{2}\right)\left(\frac{1}{x}+\frac{1}{y}\right)^{-3/2}(-1)\cdot\frac{1}{x^2}$$
$$= \frac{1}{x^3\sqrt{\frac{1}{x}+\frac{1}{y}}} - \frac{1}{4}\,\frac{1}{x^4\sqrt{\left(\frac{1}{x}+\frac{1}{y}\right)^3}}$$
$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(\frac{-1}{2y^2\sqrt{\frac{1}{x}+\frac{1}{y}}}\right) = \frac{d}{dy}\left(-\frac{1}{2}y^{-2}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1/2}\right)$$
$$= y^{-3}\left(\frac{1}{x}+\frac{1}{y}\right)^{-1/2} - \frac{1}{2}y^{-2}\left(-\frac{1}{2}\right)\left(\frac{1}{x}+\frac{1}{y}\right)^{-3/2}(-1)\cdot\frac{1}{y^2}$$
$$= \frac{1}{y^3\sqrt{\frac{1}{x}+\frac{1}{y}}} - \frac{1}{4}\,\frac{1}{y^4\sqrt{\left(\frac{1}{x}+\frac{1}{y}\right)^3}}$$

(4.3) (1) $z = xy^3 + e^{x^2y}$
$$\frac{dz}{dx} = y^3 + e^{x^2y}\cdot 2xy$$
$$\frac{dz}{dy} = 3xy^2 + e^{x^2y}\cdot x^2$$
$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(y^3 + 2xy\cdot e^{x^2y}\right)$$
$$= 2y\cdot e^{x^2y} + 2xy\cdot e^{x^2y}\cdot 2xy$$
$$(-1,1) \rightarrow 2\cdot e + (-2)\cdot e\cdot(-2)$$
$$= 6e$$
$$\frac{d^2z}{dx\,dy} = \frac{d}{dy}\left(y^3 + e^{x^2y}\cdot 2xy\right)$$
$$= 3y^2 + 2x\cdot e^{x^2y} + 2xy\cdot e^{x^2y}\cdot x^2$$
$$(-1,1) \rightarrow 3 + (-2)\cdot e + (-2)\cdot e$$
$$= 3 - 4e$$

② $z = \dfrac{x}{y(2x^2+y^2-1)} = x\,(y\cdot(2x^2+y^2-1))^{-1}$

$\dfrac{dz}{dx} =$ 'k ben 't beu!

(4.4) $z = \left(\frac{x-y}{x+y}\right)^{\rho}$

$$\frac{dz}{dx} = \rho \cdot \left(\frac{x-y}{x+y}\right)^{\rho-1} \left(\frac{1\cdot(x+y) - (x-y)\cdot 1}{(x+y)^2}\right)$$

$$= \rho \cdot \left(\frac{x-y}{x+y}\right)^{\rho-1} \left(\frac{2y}{(x+y)^2}\right)$$

$$\frac{dz}{dy} = \rho \cdot \left(\frac{x-y}{x+y}\right)^{\rho-1} \left(\frac{(-1)(x+y) - (x-y)\cdot 1}{(x+y)^2}\right)$$

$$= \rho \left(\frac{x-y}{x+y}\right)^{\rho-1} \left(\frac{-2x}{(x+y)^2}\right)$$

$$\Rightarrow x \cdot \frac{dz}{dx} + y \frac{dz}{dy} = 0$$

$$\Leftrightarrow x \cdot \rho \left(\frac{x-y}{x+y}\right)^{\rho-1}\left(\frac{2y}{(x+y)^2}\right) + y \cdot \rho \left(\frac{x-y}{x+y}\right)^{\rho-1}\left(\frac{-2x}{(x+y)^2}\right) = 0$$

$$\Leftrightarrow \frac{2\rho x y}{(x+y)^2}\left(\frac{x-y}{x+y}\right)^{\rho-1} - \frac{2\rho x y}{(x+y)^2}\left(\frac{x-y}{x+y}\right)^{\rho-1} = 0$$

$$= 0$$

(4.5) (1) $(p_A, p_B) = (1, 2)$

$U(x,y) = 2x + y$

$w = 10$

→ Budgetbeperking: $x + 2y \leq 10$

$N_0: 2x + y = 0$

$N_1: 2x + y = 1$

$N_5: 2x + y = 5$

U_{max} bij N_c: $2x + y = c$ en $(10,0) \in N_c$

$\Leftrightarrow 2 \cdot 10 + 0 = c$

$\Leftrightarrow c = 20$

$\Rightarrow U_{max} = 20$ bij $(10,0)$

(2) $(p_A, p_B) = (2,1)$
$U(x,y) = 3x+2y$
$w = 12$

→ Budgetbeperking: $2x+y \leq 12$
$N_0: 3x+2y=0$
$N_1: 3x+2y=1$
$N_5: 3x+2y=5$

U_{max} bij N_C: $3x+2y=c$ en $(0,12) \in N_C$
$3 \cdot 0 + 2 \cdot 12 = c$
$c = 24$
$\Rightarrow U_{max} = 24$ bij $(0,12)$

(3) $(p_A, p_B) = (2,5)$
$U(x,y) = 2x+5y$
$w = 10$

→ Budgetbeperking: $2x+5y \leq 10$

U_{max} bij N_C: $2x+5y=c$ en $(0,2) \in N_C$
$\Leftrightarrow 2 \cdot 0 + 5 \cdot 2 = c$
$\Leftrightarrow c = 10$
$\Rightarrow U_{max} = 10$

(4) $(p_A, p_B) = (3,2)$
$U(x,y) = x^2+y^2$
$w = 18$
→ Budgetbeperking: $3x+2y \leq 18$
U_{max} bij N_C: $x^2+y^2=c$ en $(0,9) \in N_C$
$\Leftrightarrow 0^2+9^2=c$
$\Leftrightarrow c = 81$

Hoofdstuk 5: Lokale extrema van functies van 2 veranderlijken

(5.1) ① $z = x^2 + xy + y^2 - 3x + 2$

$\frac{dz}{dx} = 2x + y - 3$

$\frac{dz}{dy} = x + 2y$

Kritieke punten

$\begin{cases} 2x+y-3=0 \\ x+2y=0 \end{cases} \Leftrightarrow \begin{cases} 2(-2y)+y-3=0 \\ x=-2y \end{cases}$

$\Leftrightarrow \begin{cases} y=-1 \\ x=2 \end{cases}$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(2x+y-3) = 2$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(x+2y) = 2$

Hessiaan: $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \rightarrow$ z bereikt een lokaal extremum in $(2,-1)$

max of min? $\frac{d^2z}{dx^2}(2,-1) = 2 > 0 \rightarrow$ lokaal minimum in $(2,-1)$
met waarde $z = 4 + (-2) + 1 - 6 + 2$
$= -1$

② $z = uv - \ln(u^2+v^2)$

Kritieke punten $\begin{cases} \frac{dz}{du} = v - \frac{2u}{u^2+v^2} = 0 \quad (1) \\ \frac{dz}{dv} = u - \frac{2v}{u^2+v^2} = 0 \quad (2) \end{cases}$

$(1) v - (2) u$: $v^2 - \frac{2uv}{u^2+v^2} - u^2 + \frac{2uv}{u^2+v^2} = v^2 - u^2$

$\Leftrightarrow v^2 = u^2$

$\Leftrightarrow v = u$ of $v = -u$

<u>1^e geval : $v = u$</u>

in (1): $u - \frac{2u}{u^2+u^2} = u - \frac{1}{u}$
$\Leftrightarrow u = \frac{1}{u}$
$\Leftrightarrow u^2 = 1$
$\Leftrightarrow u = 1$ of $u = -1$

$\Rightarrow (1,1)$ en $(-1,-1)$

<u>2^e geval : $v = -u$</u>

in (1): $-u - \frac{2u}{u^2+u^2} = -u - \frac{1}{u}$
$\Leftrightarrow u = -\frac{1}{u}$
$\Leftrightarrow u^2 = -1$

$\Rightarrow$ geen oplossingen

$$\frac{d^2x}{du^2} = \frac{d}{du}\left(v - \frac{2u}{u^2+v^2}\right) = -\left(\frac{2(u^2+v^2) - 2u \cdot 2u}{(u^2+v^2)^2}\right)$$
$$= \frac{2u^2 - 2v^2}{(u^2+v^2)^2}$$

$$\frac{d^2x}{dv^2} = \frac{d}{dv}\left(u - \frac{2v}{u^2+v^2}\right) = \frac{2v^2 - 2u^2}{(u^2+v^2)^2}$$

$$\frac{d^2x}{du\,dv} = \frac{d}{dv}\left(v - 2u\left(u^2+v^2\right)^{-1}\right) = 1 - 2u \cdot (-1) \cdot (u^2+v^2)^{-2} (2v)$$
$$= 1 + \frac{4uv}{(u^2+v^2)^2}$$

Hessiaan: (1,1)
$$\begin{vmatrix} \frac{2u^2-2v^2}{(u^2+v^2)^2} & 1 + \frac{4uv}{(u^2+v^2)^2} \\ 1 + \frac{4uv}{(u^2+v^2)^2} & \frac{2v^2-2u^2}{(u^2+v^2)^2} \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 0 - 4 = -4 < 0$$

$\Rightarrow (1,1)$ is zadelpunt van x

③ $z = 1 + x^2 - y^2$

$\frac{dz}{dx} = 2x$

$\frac{dz}{dy} = -2y$

<u>Kritieke punten</u>

$\begin{cases} 2x = 0 \\ -2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(2x) = 2$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(-2y) = -2$

Hessiaan: $\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} -4 < 0$

$\Rightarrow$ (0,0) is een zadelpunt

④ $z = 100 - x^2 - y^2$

$\frac{dz}{dx} = -2x$

$\frac{dz}{dy} = -2y$

<u>Kritieke punten</u>

$\begin{cases} -2x = 0 \\ -2y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(-2x) = -2$ Hessiaan: $\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(-2y) = -2$ max of min? $\frac{d^2z}{dx^2}(0,0) = 0$

⑤ $z = 17 + (y-x+1)^4 + y^4$

$\frac{dz}{dx} = 4(y-x+1)^3 \cdot (-1) = -4(y-x+1)^3$

$\frac{dz}{dy} = 4(y-x+1)^3 + 4y^3$

Kritieke punten

$\begin{cases} -4(y-x+1)^3 = 0 & (1) \\ 4(y-x+1)^3 + 4y^3 = 0 & (2) \end{cases}$

$(1)+(2)$: $4y^3 = 0 \Leftrightarrow y = 0$

$\begin{cases} -4(1-x)^3 = 0 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$

$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(-4(y-x+1)^3\right) = +12(y-x+1)^2$

$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(4(y-x+1)^3 + 4y^3\right) = 12(y-x+1)^2 + 12y^2$

$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(-4(y-x+1)^3\right) = -12(y-x+1)^2$

Hessiaan: $(1,0)$ $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

=> stelling geeft geen informatie

⑥ $z = 6x^2 - 4xy + 2y^2 - 12x - 4y$

$$\frac{dz}{dx} = 12x - 4y - 12$$

$$\frac{dz}{dy} = -4x + 4y - 4$$

Kritieke punten

$$\begin{cases} 12x - 4y - 12 = 0 \\ -4x + 4y - 4 = 0 \end{cases} \Leftrightarrow \begin{cases} 3x - y - 3 = 0 \\ y - x - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x + 1 \\ 3x - x - 1 - 3 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 3 \\ x = 2 \end{cases}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(12x - 4y - 12\right) = 12$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(-4x + 4y - 4\right) = 4$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}\left(12x - 4y - 12\right) = -4$$

Hessiaan: $\begin{vmatrix} 12 & -4 \\ -4 & 4 \end{vmatrix} = 4.4 \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 4.4\,(3-1) = 32 > 0$

max of min? $\frac{d^2z}{dx^2}(2,3) = 12 \Rightarrow$ lokaal minimum in $(2,3)$

met waarde

$z = 24 - 24 + 18 - 24 - 12$

$= -18$

⑦ $z = x^3 - 3bxy + y^3 \quad (b \neq 0)$

$$\frac{dz}{dx} = 3x^2 - 3by$$

$$\frac{dz}{dy} = -3bx + 3y^2$$

Kritieke punten

$$\begin{cases} x^2 - by = 0 & \Leftrightarrow y = \dfrac{x^2}{b} \\ y^2 - bx = 0 \end{cases}$$

$$\Leftrightarrow \left(\frac{x^2}{b}\right)^2 - bx = 0$$

$$\Leftrightarrow \frac{x^4}{b^2} - bx = 0$$

$$\Leftrightarrow x\left(-b + \frac{x^3}{b^2}\right) = 0$$

$$\Leftrightarrow x = 0 \quad \text{of } x^3 = b^3$$

$$\Rightarrow y = 0 \quad \wedge \ y = b$$

$\Rightarrow$ 2 kritieke punten: $(0$ en $0)$ en (b, b)

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(3x^2 - 3by\right) = 6x$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(-3bx + 3y^2\right) = 6y$$

$$\frac{d^2z}{dydx} = \frac{d}{dy}\left(3x^2 - 3by\right) = -3b$$

Hessiaan: $\begin{vmatrix} 6x & -3b \\ -3b & 6y \end{vmatrix} \underset{(0,0)}{=} \begin{vmatrix} 0 & -3b \\ -3b & 0 \end{vmatrix} = -9b^2 < 0 \Rightarrow (0,0)$ zadelpunt

$$\underset{(b,b)}{=} \begin{vmatrix} 6b & -3b \\ -3b & 6b \end{vmatrix} = 36b^2 - 9b^2 = 27b^2 > 0$$

- indien $b > 0$: $\dfrac{d^2z}{dx^2}(b,b) > 0$

  $\Rightarrow$ lokaal minimum met waarde $z = b^3 - 3b^3 + b^3 = -b^3$

- indien $b < 0$: $\dfrac{d^2z}{dx^2}(b,b) < 0$

  $\Rightarrow$ lokaal maximum met waarde $-b^3$

⑧ $z = x^2 - 4xy + y^4$

$\dfrac{dz}{dx} = 2x - 4y$

$\dfrac{dz}{dy} = -4x + 4y^3$

Kritieke punten

$\begin{cases} 2x - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 2y \\ 4y^3 - 8y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 2y \\ 4y(y^2 - 2) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \ \wedge \ x = 2\sqrt{2} \\ y = 0 \ \wedge \ y = \sqrt{2} \end{cases}$

$\Rightarrow$ 2 kritieke punten : $(0,0)$ en $(2\sqrt{2}, \sqrt{2})$ en $(-2\sqrt{2}, \sqrt{2})$

$\dfrac{d^2z}{dx^2} = \dfrac{d}{dx}(2x - 4y) = 2$

$\dfrac{d^2z}{dy^2} = \dfrac{d}{dy}(-4x + 4y^3) = 12y^2$

$\dfrac{d^2z}{dxdy} = \dfrac{d}{dy}(2x - 4y) = -4y$

Hessiaan$_{(0,0)}$: $\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$ $\Rightarrow$ stelling geeft geen informatie

Hessiaan$_{(2\sqrt{2},\sqrt{2})}$: $\begin{vmatrix} 2 & -4\sqrt{2} \\ -4\sqrt{2} & 24 \end{vmatrix} = 48 - 32 = 16 > 0$

max of min? $\dfrac{d^2z}{dx^2}(2\sqrt{2}, \sqrt{2}) = 2 > 0$

$\Rightarrow$ lokaal minimum in $(2\sqrt{2}, \sqrt{2})$

met waarde $z = 4 \cdot 2 - 4 \cdot 2 \cdot 2 + 4 = -4$

(9) $z = 8x^3 + 2xy + y^2 - 3x^2 + 1$

$$\frac{dz}{dx} = 24x^2 + 2y - 6x$$

$$\frac{dz}{dy} = 2x + 2y$$

<u>Kritieke punten</u>

$$\begin{cases} 24x^2 - 2x - 6x = 0 \\ x = -y \end{cases} \Leftrightarrow \begin{cases} 3x^2 - x = 0 \\ y = -x \end{cases}$$

$$\Leftrightarrow \begin{cases} x(3x-1) = 0 \\ y = -x \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ of } \begin{cases} x = \frac{1}{3} \\ y = -\frac{1}{3} \end{cases}$$

$\Rightarrow$ 2 kritieke punten : $(0,0)$ en $\left(\frac{1}{3}, -\frac{1}{3}\right)$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(24x^2 + 2y - 6x\right) = 48x - 6$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(2x + 2y\right) = 2$$

$$\frac{d^2z}{dxdy} = \frac{d}{dx}\left(2x + 2y\right) = 2$$

Hessiaan $(0,0)$: $\begin{vmatrix} -6 & 2 \\ 2 & 2 \end{vmatrix} = -12 - 4 = -16 < 0$
$\Rightarrow$ $(0,0)$ is een zadelpunt

Hessiaan $\left(\frac{1}{3}, -\frac{1}{3}\right)$: $\begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} = 20 - 4 = 16 > 0$

max of min? $\frac{d^2z}{dx^2}\left(\frac{1}{3}, -\frac{1}{3}\right) = 10 > 0$

$\Rightarrow$ lokaal minimum met waarde

$$z = \frac{8}{27} - \frac{2}{9} + \frac{1}{9} - \frac{3}{9} + 1$$
$$= \frac{8}{27} - \frac{6}{27} + \frac{3}{27} - \frac{9}{27} + \frac{27}{27}$$
$$= \frac{23}{27}$$

(10) $z = x + 2ey - e^{x} - e^{2y}$

$\dfrac{dz}{dx} = 1 - e^{x}$

$\dfrac{dz}{dy} = 2e - 2e^{2y}$

Kritieke punten

$\begin{cases} 1 - e^{x} \\ e^{2y} = e \end{cases} \Leftrightarrow \begin{cases} e^{x} = 1 \\ e^{2y} = e \end{cases} \Leftrightarrow \begin{cases} e^{x} = e^{0} \\ e^{2y} = e^{1} \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = \frac{1}{2} \end{cases}$

$\dfrac{d^2z}{dx^2} = \dfrac{d}{dx}\left(1 - e^{x}\right) = -e^{x}$

$\dfrac{d^2z}{dy^2} = \dfrac{d}{dy}\left(2e - 2e^{2y}\right) = -4e^{2y}$

Hessiaan $(0, \frac{1}{2})$: $\begin{vmatrix} -1 & 0 \\ 0 & -4e \end{vmatrix} = 4e > 0$

max of min ? $\dfrac{d^2z}{dx^2}(0, \frac{1}{2}) = -1 < 0$

$\Rightarrow$ lokaal maximum in $(0, \frac{1}{2})$ met waarde

$z = e - 1 - e = -1$

(11) $z = x^2 + xy + 2y^2 + 3$

$\dfrac{dz}{dx} = 2x + y$

$\dfrac{dz}{dy} = x + 4y$

Kritieke punten

$\begin{cases} 2x + y = 0 \\ x + 4y = 0 \end{cases} \Leftrightarrow \begin{cases} -7y = 0 \\ x = -4y \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$

$\dfrac{d^2z}{dx^2} = \dfrac{d}{dx}\left(2x + y\right) = 2$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}(x+4y) = 4$$

$$\frac{d^2z}{dxdy} = \frac{d}{dx}(x+4y) = 1$$

Hessiaan: $\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8-1 = 7 > 0$

max of min? $\frac{d^2z}{dx^2}(0,0) = 2 > 0$

$\Rightarrow$ lokaal minimum in $(0,0)$ met waarde $z = 3$

(12) $z = -x^2 + xy - y^2 + 2x + y$

$$\frac{dz}{dx} = -2x + y + 2$$

$$\frac{dz}{dy} = x - 2y + 1$$

Kritieke punten

$$\begin{cases} -2x+y+2 \\ x = 2y-1 \end{cases} \Leftrightarrow \begin{cases} -2(2y-1)+y+2 = 0 \\ x = 2y-1 \end{cases}$$

$$\Leftrightarrow \begin{cases} -4y+2+y+2 = 0 \\ x = 2y-1 \end{cases} \Leftrightarrow \begin{cases} y = \frac{4}{3} \\ x = \frac{5}{3} \end{cases}$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}(-2x+y+2) = -2$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}(x-2y+1) = -2$$

$$\frac{d^2z}{dxdy} = \frac{d}{dx}(x-2y+1) = 1$$

Hessiaan: $\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4-1 = 3 > 0$

max of min? $\frac{d^2z}{dx^2}\left(\frac{5}{3}, \frac{4}{3}\right) = -2 < 0$

lokaal maximum in $\left(\frac{5}{3}, \frac{4}{3}\right)$
met waarde $z = -\frac{25}{9} + \frac{20}{9} - \frac{16}{9} + \frac{10}{3} + \frac{4}{3}$
$= \frac{7}{3}$

(13) $z = ax^2 + by^2 + c$

$\frac{dz}{dx} = 2ax$

$\frac{dz}{dy} = 2by$

Kritieke punten

$\begin{cases} 2ax = 0 \\ 2by = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(2ax) = 2a$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(2by) = 2b$

Hessiaan: $\begin{vmatrix} 2a & 0 \\ 0 & 2b \end{vmatrix} = 4ab$

- indien $a > 0$ en $b > 0 \Rightarrow 4ab > 0$
  max of min? $\frac{d^2z}{dx^2}(0,0) = 2a > 0$
  $\Rightarrow$ lokaal minimum in $(0,0)$ met waarde $z = c$

- indien $a < 0$ en $b < 0 \Rightarrow 4ab > 0$
  max of min? $\frac{d^2z}{dx^2}(0,0) = 2a < 0$
  $\Rightarrow$ lokaal maximum in $(0,0)$ met waarde $z = c$

(14) $z = e^{2x} - 2x + 2y^2 + 3$

$\frac{dz}{dx} = 2e^{2x} - 2$

$\frac{dz}{dy} = 4y$

Kritieke punten

$\begin{cases} e^{2x} = 1 \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(2e^{2x}-2\right) = 4e^{2x}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}\left(4y\right) = 4$$

Hessiaan: $\underset{(0,0)}{} \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = 16 > 0$

max of min? $\frac{d^2z}{dx^2}(0,0) = 4 > 0$

$\Rightarrow$ lokaal minimum in $(0,0)$ met waarde $z = 4$

(5.2) (1) $z = 5x^2 + 6y^2 - xy$ met $x + 2y = 24$

$\mathcal{L}(x,y,\lambda) = 5x^2 + 6y^2 - xy + \lambda(x + 2y - 24)$

<u>Kritieke punten</u>

$$\frac{\partial \mathcal{L}}{\partial x} = 10x - y + \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 12y - x + 2\lambda = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + 2y - 24 = 0 \quad (3)$$

$2(1) - (2) \Leftrightarrow 20x - 2y - 12y + x = 0$

$\Leftrightarrow 21x - 14y = 0$

$\Leftrightarrow 3x - 2y = 0$

$$\begin{cases} 3x - 2y = 0 \\ x + 2y - 24 = 0 \end{cases} \Leftrightarrow \begin{cases} 3\cdot(24-2y) - 2y = 0 \\ x = 24 - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} y = 9 \\ x = 6 \end{cases}$$

$\Rightarrow 10x - y + \lambda = 0$

$\Leftrightarrow \lambda = -51$

Hessiaan: $\begin{vmatrix} \frac{d^2\mathcal{L}}{dx^2} & \frac{d^2\mathcal{L}}{dxdy} & \frac{d^2\mathcal{L}}{dxd\lambda} \\ \frac{d^2\mathcal{L}}{dxdy} & \frac{d^2\mathcal{L}}{dy^2} & \frac{d^2\mathcal{L}}{dyd\lambda} \\ \frac{d^2\mathcal{L}}{dxd\lambda} & \frac{d^2\mathcal{L}}{dyd\lambda} & 0 \end{vmatrix}$

$$\frac{d^2\mathcal{L}}{dx^2} = \frac{d}{dx}(10x - y + \lambda) = 10$$

$$\frac{d^2\mathcal{L}}{dy^2} = \frac{d}{dy}(12y - x + 2\lambda) = 12$$

$$\frac{d^2\mathcal{L}}{dxdy} = \frac{d}{dy}(10x - y + \lambda) = -1$$

$$\frac{d^2\mathcal{L}}{dxd\lambda} = \frac{d}{d\lambda}(10x - y + \lambda) = 1$$

$$\frac{d^2\mathcal{L}}{dyd\lambda} = \frac{d}{d\lambda}(12y - x + 2\lambda) = 2$$

Hessiaan $(6, 9, -51)$

$$\begin{vmatrix} 10 & -1 & 1 \\ -1 & 12 & 2 \\ 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 10 & -21 & 1 \\ -1 & 14 & 2 \\ 1 & 0 & 0 \end{vmatrix} = -42 - 14 < 0$$

$\Rightarrow$ gebonden minimum in $(6, 9)$ met waarde $5 \cdot 6^2 + 6 \cdot 9^2 - 6 \cdot 9$

$$= 6(30 + 81 - 9)$$

$$= 6 \cdot 102 = 612$$

② $z = 12xy - x^2 - 3y^2$ met $x + y = 16$

$\mathcal{L}(x, y, \lambda) = 12xy - x^2 - 3y^2 + \lambda(x + y - 16)$

Kritieke punten

$\frac{d\mathcal{L}}{dx} = 12y - 2x + \lambda = 0$ (1) $\quad$ (1)-(2): $12y - 2x - 12x + 6y = 0$

$\frac{d\mathcal{L}}{dy} = 12x - 6y + \lambda = 0$ (2) $\quad \Leftrightarrow 18y - 14x = 0$

$\quad \Leftrightarrow 9y - 7x = 0$

$\frac{d\mathcal{L}}{d\lambda} = x + y - 16 = 0$ (3)

$$\begin{cases} 9y - 7x \\ x = 16 - y \end{cases} \Leftrightarrow \begin{cases} 9y - 112 + 7y \\ x = 16 - y \end{cases} \Leftrightarrow \begin{cases} y = 7 \\ x = 9 \end{cases}$$

$\Rightarrow 12 \cdot 7 - 2 \cdot 9 + \lambda = 0$

$\Rightarrow 84 - 18 + \lambda = 0$

$\Rightarrow \lambda = 66$

$$\frac{d^2\mathcal{L}}{dx^2} = \frac{d}{dx}\left(12y - 2x + \lambda\right) = -2$$
$$\frac{d^2\mathcal{L}}{dy^2} = \frac{d}{dy}\left(12x - 6y + \lambda\right) = -6$$
$$\frac{d^2\mathcal{L}}{dxdy} = \frac{d}{dy}\left(12y - 2x + \lambda\right) = 12$$
$$\frac{d^2\mathcal{L}}{dxd\lambda} = \frac{d}{d\lambda}\left(12y - 2x + \lambda\right) = 1$$
$$\frac{d^2\mathcal{L}}{dyd\lambda} = \frac{d}{d\lambda}\left(12x - 6y + \lambda\right) = 1$$

Hessiaan $(9,7,66)$: $\begin{vmatrix} -2 & 12 & 1 \\ 12 & -6 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 14 & 1 \\ 12 & -18 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 14 & 1 \\ -18 & 1 \end{vmatrix} = 14 + 18 = 32 > 0$

$\Rightarrow$ gebonden maximum in $(9,7)$ met waarde $z = 12 \cdot 9 \cdot 7 - 81 - 3 \cdot 49$
$= 756 - 81 - 147$
$= 528$

③ $z = x^2 + y^2 - 2xy$, met $x^2 + y^2 = 50$

$\mathcal{L}(x, y, \lambda) = x^2 + y^2 - 2xy + \lambda(x^2 + y^2 - 50)$

$\frac{d\mathcal{L}}{dx} = 2x - 2y + 2x\lambda = 0 \quad (1)$ $\qquad (1)+(2): 2x\lambda + 2y\lambda = 0$

$\frac{d\mathcal{L}}{dy} = 2y - 2x + 2y\lambda = 0 \quad (2)$ $\qquad \Leftrightarrow 2\lambda(x+y) = 0$

$\frac{d\mathcal{L}}{d\lambda} = x^2 + y^2 - 50 = 0 \quad (3)$ $\qquad \Leftrightarrow \lambda = 0$ of $x = -y$

- 1e geval: $\lambda = 0$

in (1): $2x - 2y = 0 \Leftrightarrow x = y$

in (3): $x^2 + x^2 - 50 = 0 \Leftrightarrow x^2 = 25$

$\Leftrightarrow x = 5$ of $x = -5$

$y = 5$ of $y = -5$

$\Rightarrow$ 2 kritieke punten: $(5,5,0)$ en $(-5,-5,0)$

• 2^e geval: $y = -x$

in (3): $x^2 + (-x)^2 = 50$

$\Leftrightarrow x^2 = 25$

$\Leftrightarrow x = 5$ of $x = -5$

$y = -5$ of $y = 5$

in (1): $2\lambda x = -2x + 2y$

$\Leftrightarrow \lambda = \frac{-x+y}{x}$

$\Rightarrow x = 5;\ y = -5 \rightarrow \lambda = \frac{-5-5}{5} = -2$

$\Rightarrow x = -5;\ y = 5 \rightarrow \lambda = \frac{5+5}{-5} = -2$

$\Rightarrow$ 2 kritieke punten: $(5,-5,-2)$ en $(-5,5,-2)$

$\frac{d^2\mathcal{L}}{dx^2} = \frac{d}{dx}(2x - 2y + 2\lambda x) = 2 + 2\lambda$

$\frac{d^2\mathcal{L}}{dy^2} = \frac{d}{dy}(2y - 2x + 2\lambda y) = 2 + 2\lambda$

$\frac{d^2\mathcal{L}}{dxdy} = \frac{d}{dy}(2x - 2y + 2\lambda x) = -2$

$\frac{d^2\mathcal{L}}{dxd\lambda} = \frac{d}{d\lambda}(2x - 2y + 2\lambda x) = 2x$

$\frac{d^2\mathcal{L}}{dyd\lambda} = \frac{d}{d\lambda}(2y - 2x + 2\lambda y) = 2y$

Hessiaan: $\begin{vmatrix} 2+2\lambda & -2 & 2x \\ -2 & 2+2\lambda & 2y \\ 2x & 2y & 0 \end{vmatrix}$ $\quad H_{(5,5,0)}: \begin{vmatrix} 2 & -2 & 10 \\ -2 & 2 & 10 \\ 10 & 10 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -4 & 10 \\ -2 & 4 & 10 \\ 10 & 0 & 0 \end{vmatrix}$

$= 10.(-40-40) = -800 < 0$

$H_L(-5,-5,0): \begin{vmatrix} 2 & -2 & -10 \\ -2 & 2 & -10 \\ -10 & -10 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -4 & -10 \\ -2 & 4 & -10 \\ -10 & 0 & 0 \end{vmatrix} = -10(40+40)$

$= -800 < 0$

$\Rightarrow$ gebonden minimum in $(5,5)$ met waarde 0

$(-5,-5)$

$$H_L(5,-5,-2) = \begin{vmatrix} -2 & -2 & 10 \\ -2 & -2 & -10 \\ 10 & -10 & 0 \end{vmatrix} = \begin{vmatrix} -2 & -4 & 10 \\ -2 & -4 & -10 \\ 10 & 0 & 0 \end{vmatrix} = 10(40+40) = 800 > 0$$

$$H_L(-5,5,-2) = \begin{vmatrix} -2 & -2 & -10 \\ -2 & -2 & 10 \\ -10 & 10 & 0 \end{vmatrix} = \begin{vmatrix} -2 & -4 & -10 \\ -2 & -4 & 10 \\ -10 & 0 & 0 \end{vmatrix} = -10(-40-40) = 800 > 0$$

$\Rightarrow$ gebonden maximum in $(-5,5)$ met waarde 100
$(5,-5)$

④ $z = x^2 - 10y^2$, met $x - y = 18$

$L(x,y,\lambda) = x^2 - 10y^2 + \lambda(x - y - 18)$

$\frac{dz}{dx} = 2x + \lambda = 0 \quad (1)$ $\quad (1)+(2): 2x - 20y = 0$

$\frac{dz}{dy} = -20y - \lambda = 0 \quad (2)$ $\quad \begin{cases} 2x - 20y = 0 \\ x = 18 + y \end{cases} \Leftrightarrow \begin{cases} 2(18+y) - 20y = 0 \\ x = 18 + y \end{cases}$

$\frac{dz}{d\lambda} = x - y - 18 = 0 \quad (3)$

$\Leftrightarrow \begin{cases} y = 2 \\ x = 20 \end{cases}$

$\Rightarrow$ in (1): $\lambda = -40$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(2x + \lambda) = 2$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(-20y - \lambda) = -20$

$\frac{d^2z}{dxdy} = \frac{d}{dy}(2x + \lambda) = 0$

$\frac{d^2z}{dxd\lambda} = \frac{d}{d\lambda}(2x + \lambda) = 1$

$\frac{d^2z}{dyd\lambda} = \frac{d}{d\lambda}(-20y - \lambda) = -1$

Hessiaan: $\begin{vmatrix} 2 & 0 & 1 \\ 0 & -20 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 1 \\ 0 & -20 & -1 \\ 1 & 0 & 0 \end{vmatrix} = (-2) + 20 = 18 > 0$

$\Rightarrow$ gebonden minimum in $(20,2)$ met waarde $400 - 40 = 360$

⑥ $z = x + y$, met $x^2+y^2=1$

$\mathcal{L}(x,y,\lambda) = x+y+\lambda(x^2+y^2-1)$

$\frac{dz}{dx} = 1+2\lambda x = 0 \quad (1)$ $\qquad (1)-(2):\ 2\lambda x - 2\lambda y$

$\frac{dz}{dy} = 1+2\lambda y = 0 \quad (2)$ $\qquad \Leftrightarrow 2\lambda(x-y)=0$

$\qquad \Leftrightarrow \lambda = 0$ of $x=y$

$\frac{dz}{d\lambda} = x^2+y^2-1 = 0 \quad (3)$

- 1e geval: $\lambda = 0$ → kan niet
- 2e geval: $x = y \Leftrightarrow x^2+x^2=1$

$\Leftrightarrow x^2 = \frac{1}{2}$

$\Leftrightarrow x = \frac{1}{\sqrt{2}}$ of $x = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ 2 kritieke punten: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ en $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(1+2\lambda x) = 2\lambda$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(1+2\lambda y) = 2\lambda$ $\qquad$ Hessiaan: $\begin{vmatrix} 2\lambda & 0 & 2x \\ 0 & 2\lambda & 2y \\ 2x & 2y & 0 \end{vmatrix}$

$\frac{d^2z}{dxdy} = \frac{d}{dy}(1+2\lambda x) = 0$

$\frac{d^2z}{dxd\lambda} = \frac{d}{d\lambda}(1+2\lambda x) = 2x$

$\frac{d^2z}{dyd\lambda} = \frac{d}{d\lambda}(1+2\lambda y) = 2y$

$H_{\mathcal{L}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-\sqrt{2}}{2}\right) = \begin{vmatrix} -\sqrt{2} & 0 & \frac{2}{\sqrt{2}} \\ 0 & -\sqrt{2} & \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \end{vmatrix} = \begin{vmatrix} -\sqrt{2} & \sqrt{2} & 2/\sqrt{2} \\ 0 & -\sqrt{2} & \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & 0 & 0 \end{vmatrix} = \frac{2}{\sqrt{2}}(2+2) = \frac{8}{\sqrt{2}}$

$\Rightarrow$ gebonden max, waarde $\frac{2}{\sqrt{2}}$

$H_{\mathcal{L}}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{2}\right) = \begin{vmatrix} \sqrt{2} & 0 & -2/\sqrt{2} \\ 0 & \sqrt{2} & -2/\sqrt{2} \\ -\frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} & 0 \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{2} & -2/\sqrt{2} \\ 0 & \sqrt{2} & -2/\sqrt{2} \\ -2/\sqrt{2} & 0 & 0 \end{vmatrix} = -\frac{2}{\sqrt{2}}(2+2) = \frac{-8}{\sqrt{2}}$

$\Rightarrow$ gebonden min, waarde $-2/\sqrt{2}$

⑥ $z = x^3y^3$, met $2x+3y=48$

$L(x,y,\lambda) = x^3y^3 + \lambda(2x+3y-48)$

$\frac{dz}{dx} = 3x^2y^3 + 2\lambda \quad (1)$ $\qquad 3(1)-2(2): 9x^2y^3 - 6y^2x^3$

$\frac{dz}{dy} = 3y^2x^3 + 3\lambda \quad (2)$ $\qquad \Leftrightarrow 3x^2y^2(3y-2x)=0$

$\Leftrightarrow x=0 \wedge y=0 \wedge 3y-2x=0$

$\frac{dz}{d\lambda} = 2x+3y-48 \quad (3)$

$y=16 \quad x=24 \quad x=12;\ y=8$

$\lambda=0 \quad \lambda=0 \quad \lambda=110592$

$\Rightarrow$ 3 kritieke punten: $(0,16,0)$; $(24,0,0)$; $(12,8,110592)$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(3x^2y^3+2\lambda) = 6xy^3$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(3y^2x^3+3\lambda) = 6yx^3$

$\frac{d^2z}{dxdy} = \frac{d}{dy}(3x^2y^3+2\lambda) = 9y^2x^2$

$\frac{d^2z}{dxd\lambda} = \frac{d}{d\lambda}(3x^2y^3+2\lambda) = 2$

$\frac{d^2z}{dyd\lambda} = \frac{d}{d\lambda}(3y^2x^3+3\lambda) = 3$

Hessiaan: $\begin{vmatrix} 6xy^3 & 9y^2x^2 & 2 \\ 9y^2x^2 & 6yx^3 & 3 \\ 2 & 3 & 0 \end{vmatrix}$

$H_L(0,16,0): \begin{vmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 0 \end{vmatrix} = 0$

$H_L(24,0,0): \begin{vmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 0 \end{vmatrix} = 0$

$H_L(12,8,\ldots): \begin{vmatrix} 72\cdot 8^3 & 9\cdot 144\cdot 64 & 2 \\ 9\cdot 64\cdot 144 & 48\cdot 12^3 & 3 \\ 2 & 3 & 0 \end{vmatrix} \ldots$

(7) $z = x^2 + y^2$, met $3x + 4y = 12$

$L(x, y, \lambda) = x^2 + y^2 + \lambda(3x + 4y - 12)$

$\frac{dz}{dx} = 2x + 3\lambda = 0$ (1) $\quad 4.(1) - 3(2)$: $8x - 6y = 0$

$\frac{dz}{dy} = 2y + 4\lambda = 0$ (2)

$\frac{dz}{d\lambda} = 3x + 4y - 12 = 0$ (3)

$\begin{cases} x = \frac{3}{4}y \\ 3(\frac{3}{4}y) + 4y - 12 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{3}{4}y \\ \frac{9}{4}y + 4y - 12 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = \frac{3}{4}y \\ 9y + 16y - 48 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{36}{25} \\ y = \frac{48}{25} \end{cases}$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(2x + 3\lambda) = 2$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(2y + 4\lambda) = 2$

$\frac{d^2z}{dxdy} = \frac{d}{dy}(2x + 3\lambda) = 0$

$\frac{d^2z}{dxd\lambda} = \frac{d}{d\lambda}(2x + 3\lambda) = 3$

$\frac{d^2z}{dyd\lambda} = \frac{d}{d\lambda}(2y + 4\lambda) = 4$

Hessiaan: $\begin{vmatrix} 2 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 4 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$

$= 2(-16) + 3(-6)$

$= -50$

$\Rightarrow$ gebonden minimum in $\left(\frac{36}{25}, \frac{48}{25}\right)$ met waarde $\frac{1296 + 2304}{625}$

$= \frac{3600}{625}$

⑧ $z = x^2+y^2$, met $xy=5$

$L(x,y,\lambda) = x^2+y^2+\lambda(xy-5)$

$\frac{dz}{dx} = 2x+\lambda y = 0$ $\quad y\cdot(1) - x\cdot(2): \lambda y^2 - \lambda x^2 = 0$

$\frac{dz}{dy} = 2y+\lambda x = 0$ $\quad \Leftrightarrow \lambda(y-x)(y+x) = 0$

$\frac{dz}{d\lambda} = xy-5=0$

- 1e geval : $\lambda = 0$

  in (1): $2x = 0 \Leftrightarrow x=0$

  in (3) $\qquad y=0$

- 2e geval: $y=x$

  in (3) : $x^2=5$

  $x = \pm\sqrt{5} \rightarrow y = \pm\sqrt{5}$

  $\Rightarrow$ 2 kritieke punten : $(\sqrt{5},\sqrt{5},-2)$ en $(-\sqrt{5},-\sqrt{5},-2)$

- 3e geval: $y=-x$

  $x = \pm\sqrt{-5} \rightarrow y = \pm\sqrt{-5}$

  $\Rightarrow$ 2 kritieke punten : $(\sqrt{-5},\sqrt{-5},2)$ en $(-\sqrt{-5},-\sqrt{-5},2)$

$\frac{d^2z}{dx^2} = \frac{d}{dx}(2x+\lambda y) = 2$

$\frac{d^2z}{dy^2} = \frac{d}{dy}(2y+\lambda x) = 2$

$\frac{d^2z}{dxdy} = \frac{d}{dy}(2x+\lambda y) = \lambda$

$\frac{d^2z}{dxd\lambda} = \frac{d}{d\lambda}(2x+\lambda y) = y$

$\frac{d^2z}{dyd\lambda} = \frac{d}{d\lambda}(2y+\lambda x) = x$

zelf verder uitwerken, te laat, ben het beu

⑨ $z = e^x + e^y - x - y$, met $x+y=2$

$L(x,y,\lambda) = e^x + e^y - x - y + \lambda(x+y-2)$

$\frac{\partial L}{\partial x} = e^x - 1 + \lambda = 0 \quad (1)$ $\qquad (1)-(2): e^x - e^y = 0$

$\frac{\partial L}{\partial y} = e^y - 1 + \lambda = 0 \quad (2)$ $\qquad \Leftrightarrow e^x = e^y$

$\frac{\partial L}{\partial \lambda} = x+y-2 = 0 \quad (3)$ $\qquad \Leftrightarrow x = y$

$$\begin{cases} x = y \\ 2x-2=0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$\Rightarrow \lambda = 1 - e$

$\Rightarrow$ kritiek punt $(1, 1, 1-e)$

$\frac{\partial^2 L}{\partial x^2} = \frac{\partial}{\partial x}(e^x - 1 + \lambda) = e^x$

$\frac{\partial^2 L}{\partial y^2} = \frac{\partial}{\partial y}(e^y - 1 + \lambda) = e^y$

$\frac{\partial^2 L}{\partial x \partial y} = \frac{\partial}{\partial y}(e^x - 1 + \lambda) = 0$

$\frac{\partial^2 L}{\partial x \partial \lambda} = \frac{\partial}{\partial \lambda}(e^x - 1 + \lambda) = 1$

$\frac{\partial^2 L}{\partial y \partial \lambda} = \frac{\partial}{\partial \lambda}(e^y - 1 + \lambda) = 1$

Hessiaan: $\begin{vmatrix} e & 0 & 1 \\ 0 & e & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} e & -e & 1 \\ 0 & e & 1 \\ 1 & 0 & 0 \end{vmatrix} = -e - e = -2e < 0$

$\Rightarrow$ gebonden minimum in $(1,1)$ met waarde $e + e - 2$

$= 2e - 2$

(10) $z = 3y^2 - 8y^3 - 2xy - x^2 - 1$, met $x+y=0$

$L(x,y,\lambda) = 3y^2 - 8y^3 - 2xy - x^2 - 1 + \lambda(x+y)$

$$\frac{dz}{dx} = -2y - 2x + \lambda = 0$$

$$\frac{dz}{dy} = 6y - 24y^2 - 2x + \lambda = 0$$

$$\frac{dz}{d\lambda} = x + y = 0$$

$(1)-(2)$: $-2y - 2x - 6y + 24y^2 + 2x = 0$

$\Leftrightarrow 24y^2 - 8y = 0$

$\Leftrightarrow \cancel{8}y(3y-1) = 0$

$\Leftrightarrow y = 0 \;\wedge\; y = \frac{1}{3}$

$x = 0 \qquad x = -\frac{1}{3}$

$\lambda = 0 \qquad \lambda = 0$

$\Rightarrow$ 2 kritieke punten: $(0,0,0)$ en $\left(-\frac{1}{3}, \frac{1}{3}, 0\right)$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}(-2y - 2x + \lambda) = -2$$

$$\frac{d^2z}{dy^2} = \frac{d}{dy}(6y - 24y^2 - 2x + \lambda) = 6 - 48y$$

$$\frac{d^2z}{dxdy} = \frac{d}{dy}(-2y - 2x + \lambda) = -2$$

$$\frac{d^2z}{dxd\lambda} = \frac{d}{d\lambda}(-2y - 2x + \lambda) = 1$$

$$\frac{d^2z}{dyd\lambda} = \frac{d}{d\lambda}(6y - 24y^2 - 2x + \lambda) = 1$$

Hessiaan: $\begin{vmatrix} -2 & -2 & 1 \\ -2 & 6-48y & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$H_L(0,0,0) = \begin{vmatrix} -2 & -2 & 1 \\ -2 & 6 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 1 \\ -2 & 8 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -8 < 0$ $\Rightarrow$ gebonden min in $(0,0)$ met waarde -1

$H_L\left(-\frac{1}{3}, \frac{1}{3}, 0\right) = \begin{vmatrix} -2 & -2 & 1 \\ -2 & -10 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 1 \\ -2 & -8 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 8 > 0$ $\Rightarrow$ gebonden max in $\left(-\frac{1}{3}, \frac{1}{3}\right)$ met waarde $-\frac{23}{27}$

(5.3) $p_A = 36 - 3q_A$

$p_B = 40 - 5q_B$

totale kostenfunctie: $C = q_A^2 + 2q_Aq_B + 3q_B^2$

Gevr: winst monopolist maximaliseren: W_{max}

Stel $q_A = x$ en $q_B = y$

$$
\begin{aligned}
W(q_A, q_B) &= p_A \cdot q_A + p_B \cdot q_B - C \\
&= (36-3x)\,x + (40-5y)\,y - x^2 - 2xy - 3y^2 \\
&= 36x - 3x^2 + 40y - 5y^2 - x^2 - 2xy - 3y^2 \\
&= -4x^2 - 8y^2 - 2xy + 36x + 40y
\end{aligned}
$$

$$\frac{dW}{dx} = -8x - 2y + 36 = 0 \qquad \Leftrightarrow \begin{cases} -4x - y + 18 = 0 \\ -8y - x + 20 = 0 \end{cases}$$

$$\frac{dW}{dy} = -16y - 2x + 40 = 0$$

$$\Leftrightarrow \begin{cases} -4(20-8y) - y + 18 = 0 \\ x = -8y + 20 \end{cases} \Leftrightarrow \begin{cases} -80 + 32y - y + 18 = 0 \\ x = -8y + 20 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 2 \\ x = 4 \end{cases}$$

$$\frac{d^2W}{dx^2} = \frac{d}{dx}(-8x - 2y + 36) = -8$$

$$\frac{d^2W}{dy^2} = \frac{d}{dy}(-16y - 2x + 40) = -16$$

$$\frac{d^2W}{dxdy} = \frac{d}{dy}(-8x - 2y + 36) = -2$$

Hessiaan: $\begin{vmatrix} -8 & -2 \\ -2 & -16 \end{vmatrix} = 128 - 4 = 124 > 0$

max of min? $\frac{d^2W}{dx^2}(4,2) = -8 < 0$

$\Rightarrow$ lokaal maximum in $(4,2)$ met waarde $\underbrace{112}_{W_{max}}$

→ vul zelf in in functies

(5.4) $q = 2q_A + q_B$

$p = 8 - q$

$\left.\begin{array}{l} k_A = q_A + 3 \\ k_B = q_B + 1 \end{array}\right\}$ per eenheid

gebruik x en y

$$\begin{aligned} W(q_A, q_B) &= p\,q - k_A\,q_A - k_B\,q_B \\ &= (8 - 2x - y)\cdot(2x + y) - (x + 3)\,x - (y + 1)\cdot y \\ &= 16x + 8y - 4x^2 - 2xy - 2xy - y^2 - x^2 - 3x - y^2 - y \\ &= -5x^2 - 2y^2 - 4xy + 13x + 7y \end{aligned}$$

$\dfrac{dW}{dx} = -10x - 4y + 13 = 0 \ (1)$ $\quad (1) - (2): -10x + 13 + 4x - 7$

$\qquad\qquad\qquad\qquad\qquad\qquad \Leftrightarrow -6x + 6 = 0$

$\dfrac{dW}{dy} = -4y - 4x + 7 = 0 \ (2)$ $\quad \Leftrightarrow 6(1 - x) = 0$

$\begin{cases} x = 1 \\ y = \frac{3}{4} \end{cases}$

$\dfrac{d^2W}{dx^2} = \dfrac{d}{dx}(-10x - 4y + 13) = -10$

$\dfrac{d^2W}{dy^2} = \dfrac{d}{dy}(-4y - 4x + 7) = -4$

$\dfrac{d^2W}{dxdy} = \dfrac{d}{dy}(-10x - 4y + 13) = -4$

Hessiaan: $\begin{vmatrix} -10 & -4 \\ -4 & -4 \end{vmatrix} = 40 - 16 = 24 > 0$

max of min ? $\dfrac{d^2W}{dx^2}\left(1, \frac{3}{4}\right) = -10$

$\Rightarrow$ lokaal maximum in $\left(1, \frac{3}{4}\right)$ met waarde

$$\begin{aligned} x &= -5 - 2\cdot\frac{9}{16} - 4\cdot\frac{3}{4} + 13 + 7\cdot\frac{3}{4} \\ &= -5 - \frac{9}{8} - 3 + 13 + \frac{21}{4} \\ &= \frac{73}{8} \rightarrow W_{max} \end{aligned}$$

(5.5) $q = 8q_A q_B$

$\left.\begin{array}{l} p_A = 10 \\ p_B = 20 \end{array}\right\}$ per eenheid

① q_A, q_B? zodat q maximaal is en budget $= 35$

→ budgetrestrictie (= nvw)

$$10q_A + 20q_B = 35$$

$$g(q_A, q_B) = 10q_A + 20q_B - 35 = 0$$

$$L(q_A, q_B, \lambda) = 8q_A q_B + \lambda(10q_A + 20q_B - 35)$$
$$= 8xy + \lambda(10x + 20y - 35)$$

$$\frac{dL}{dx} = 8y + 10\lambda = 0 \quad (1)$$

$$\frac{dL}{dy} = 8x + 20\lambda = 0 \quad (2)$$

$$\frac{dL}{d\lambda} = 10x + 20y - 35 = 0 \quad (3)$$

$2(1) - (2)$: $16y - 8x = 0$
$\Leftrightarrow 2y = x$
in (3): $10(2y) + 20y = 35$
$\Leftrightarrow 40y = 35$
$\Leftrightarrow y = \frac{7}{8} \Rightarrow x = \frac{7}{4}$

$$\Rightarrow 8 \cdot \frac{7}{8} + 10\lambda = 0 \Leftrightarrow \lambda = -\frac{7}{10}$$

$$\frac{d^2L}{dx^2} = \frac{d}{dx}(8y + 10\lambda) = 0$$

$$\frac{d^2L}{dy^2} = \frac{d}{dy}(8x + 20\lambda) = 0$$

$$\frac{d^2L}{dxdy} = \frac{d}{dy}(8y + 10\lambda) = 8$$

$$\frac{d^2L}{dxd\lambda} = \frac{d}{d\lambda}(8y + 10\lambda) = 10$$

$$\frac{d^2L}{dyd\lambda} = \frac{d}{d\lambda}(8x + 20\lambda) = 20$$

Hessiaan: $\begin{vmatrix} 0 & 8 & 10 \\ 8 & 0 & 20 \\ 10 & 20 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 10 \\ 8 & -16 & 20 \\ 10 & 0 & 0 \end{vmatrix} = 10(160 + 160) = 320 > 0$

$\Rightarrow q_{max} = 8 \cdot \frac{7}{4} \cdot \frac{7}{8} = \frac{49}{4}$

Hoofdstuk 6: Determinanten

(6.1)

(1)
$$\begin{vmatrix} 3 & 2 & 4 & 2 \\ 2 & 1 & 5 & 3 \\ 1 & 5 & -2 & 2 \\ -3 & -2 & -4 & 1 \end{vmatrix} \underset{R_1+R_4}{=} \begin{vmatrix} 0 & 0 & 0 & 3 \\ 2 & 1 & 5 & 3 \\ 1 & 5 & -2 & 2 \\ -3 & -2 & -4 & 1 \end{vmatrix} = -3 \begin{vmatrix} 2 & 1 & 5 \\ 1 & 5 & -2 \\ -3 & -2 & -4 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 0 & -9 & 9 \\ 1 & 5 & -2 \\ 0 & 13 & -10 \end{vmatrix} = -3\cdot(-1) \begin{vmatrix} -9 & 9 \\ 13 & -10 \end{vmatrix}$$

$$= 3\cdot 9 \begin{vmatrix} -1 & 1 \\ 13 & -10 \end{vmatrix}$$

$$= 3\cdot 9\,(10-13) = -81$$

(2)
$$\begin{vmatrix} 1 & 3 & 4 & 5 \\ 0 & 6 & 3 & -2 \\ 0 & 0 & 8 & -7 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 6 & 3 & -2 \\ 0 & 8 & -7 \\ 0 & 0 & 1 \end{vmatrix} = 6 \begin{vmatrix} 8 & -7 \\ 0 & 1 \end{vmatrix} = 6\,(8-0) = 48$$

(3)
$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & -2 & 3 \\ 8 & -9 & 0 & 5 & -2 \\ 12 & 7 & 0 & 1 & 7 \\ 3 & -10 & 0 & 3 & -7 \end{vmatrix} = \begin{vmatrix} 4 & 1 & -2 & 3 \\ -9 & 0 & 5 & -2 \\ 7 & 0 & 1 & 7 \\ -10 & 0 & 3 & -7 \end{vmatrix} = (-1) \begin{vmatrix} -9 & 5 & -2 \\ 7 & 1 & 7 \\ -10 & 3 & -7 \end{vmatrix}$$

$$\underset{K_1-K_3}{=} \begin{vmatrix} -7 & 5 & -2 \\ 0 & 1 & 7 \\ -3 & 3 & -7 \end{vmatrix} \underset{K_3-7K_1}{=} - \begin{vmatrix} -7 & 5 & -37 \\ 0 & 1 & 0 \\ -3 & 3 & -28 \end{vmatrix}$$

$$= (-1)\big((-7)(-28) - (-3)(-37)\big)$$
$$= (-1)(196 - 111)$$
$$= -85$$

(4) $\begin{vmatrix} 3 & -1 & 2 & 1 \\ 4 & 2 & 0 & -3 \\ -2 & 1 & -3 & 2 \\ 1 & 3 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 13 & -1 & 6 & -3 \\ -8 & 3 & -7 & 2 \\ -11 & 7 & -9 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 13 & -1 & 6 \\ -8 & 3 & -7 \\ -11 & 7 & -9 \end{vmatrix}$

$= (-1)\left[13\begin{vmatrix} 3 & -7 \\ 7 & -9 \end{vmatrix} + \begin{vmatrix} -8 & -7 \\ -11 & -9 \end{vmatrix} + 6\begin{vmatrix} -8 & 3 \\ -11 & 7 \end{vmatrix} \right]$

$= -13(-27+49) - (72-77) - 6(-56+33)$

$= -13 \cdot 22 - (-5) - 6 \cdot (-23)$

$= -286 + 5 + 138$

$= -156 + 5 + 8$

$= -143$

(5) $\begin{vmatrix} 4 & 1 & -2 & 3 \\ -1 & 2 & 1 & 4 \\ 3 & -1 & 3 & 4 \\ 2 & 3 & -3 & 2 \end{vmatrix} \underset{R_3 - R_4}{=} \begin{vmatrix} 4 & 1 & -2 & 3 \\ -1 & 2 & 1 & 4 \\ 5 & 2 & 0 & 6 \\ 2 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 0 & 11 \\ -1 & 2 & 1 & 4 \\ 5 & 2 & 0 & 6 \\ -1 & 9 & 0 & 14 \end{vmatrix}$

$= (-1)\begin{vmatrix} 2 & 5 & 11 \\ 5 & 2 & 6 \\ -1 & 9 & 14 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 11 \\ 5 & 2 & 6 \\ 1 & -9 & -14 \end{vmatrix} = \begin{vmatrix} 0 & 23 & 39 \\ 0 & 47 & 76 \\ 1 & -9 & -14 \end{vmatrix}$

$= \begin{vmatrix} 23 & 39 \\ 47 & 76 \end{vmatrix} = 23 \cdot 76 - 47 \cdot 39$

$= 1748 - 1833$

$= -85$

(6) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 3 \\ 3 & 8 & 12 & 2 \\ 4 & 16 & 24 & 1 \end{vmatrix} = 2 \cdot 3 \begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 3 \\ 3 & 4 & 4 & 2 \\ 4 & 8 & 8 & 1 \end{vmatrix} = 0$

(7)
$$\begin{vmatrix} -2 & 4 & 1 & 3 \\ 1 & -2 & 0 & 4 \\ 0 & 1 & -3 & 0 \\ 4 & 3 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 4 & 1 & 3 \\ 1 & -2 & 0 & 4 \\ -6 & 13 & 0 & 9 \\ 4 & 3 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 4 \\ -6 & 13 & 9 \\ 4 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -6 & 1 & 33 \\ 4 & 11 & -17 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 33 \\ 11 & -17 \end{vmatrix} = -17 - 33.11 = -17 - 363 = -380$$

(8)
$$\begin{vmatrix} -1 & 2 & 3 & -2 \\ 4 & -1 & -2 & 2 \\ -3 & 1 & 2 & -1 \\ 2 & 4 & -1 & 3 \end{vmatrix} \underset{R_2+R_3}{=} \begin{vmatrix} -1 & 2 & 3 & -2 \\ 1 & 0 & 0 & 1 \\ -3 & 1 & 2 & -1 \\ 2 & 4 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 2 & -1 \\ -1 & 4 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ -1 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 8 \\ 6 & 2 \end{vmatrix} = 10 - 48 = -38$$

(9)
$$\begin{vmatrix} 1 & 4 & 8 \\ -2 & 1 & 5 \\ -3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 8 \\ 0 & 9 & 21 \\ 0 & 14 & 28 \end{vmatrix} = \begin{vmatrix} 9 & 21 \\ 14 & 28 \end{vmatrix} = 3.14 \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix}$$

$$= 3.14\,(6-7) = -42$$

(10)
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix} =$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c+a-b-a) = (b-a)(c-a)(c-b)$$

(11)
$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 2 & -2 \\ 2 & 4 & 2 & 1 \\ 3 & 1 & 5 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & -6 \\ 2 & 4 & 4 & -3 \\ 3 & 1 & 8 & -9 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -6 \\ 4 & 4 & -3 \\ 1 & 8 & -9 \end{vmatrix}$$

$$= 4.(-3) \begin{vmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 4.(-3) \begin{vmatrix} 3 & 1 & 2 \\ 1 & 0 & -1 \\ -5 & 0 & -1 \end{vmatrix}$$

$$= 4.(-3)(-1) \begin{vmatrix} 1 & -1 \\ -5 & -1 \end{vmatrix}$$

$$= 4(-3).(-1)(-1-5)$$

$$= -72$$

(12)
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ b^2+ab+a^2 & c^2+ac+a^2 & d^2+ad+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 0 & 0 \\ b+a & c-b & d-b \\ b^2+ab+a^2 & c^2+ac-b^2-ab & d^2+ad-b^2-ab \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} c-b & d-b \\ c^2+ac-b^2-ab & d^2+ad-b^2-ab \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b) \begin{vmatrix} 1 & 1 \\ c+b+a & d+b+a \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d+b+a-c-b-a)$$

(6-2)

$$\begin{vmatrix} x & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 1 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & x \end{vmatrix} = \begin{vmatrix} x+4 & x+4 & x+4 & x+4 & x+4 \\ 1 & x & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 1 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+4)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 1 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & x \end{vmatrix} = (x+4)(x-1)^4$$

det 0 $\Leftrightarrow$ $x+4=0$ of $(x-1)^4=0$

$\Leftrightarrow x=-4$ $\qquad$ $\Leftrightarrow x=1$

Hoofdstuk 7: Matrices

(7.1)

(1) $\begin{bmatrix} 4 & -1 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 9 & 7 \end{bmatrix}$

(2) $\begin{bmatrix} 8 & 3 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & -8 \end{bmatrix}$

(3) $3 \cdot \begin{bmatrix} 4 & -1 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ 18 & 27 \end{bmatrix}$

(4) $4 \cdot \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix} + 2\begin{bmatrix} 8 & 3 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 12 & -8 \end{bmatrix} + \begin{bmatrix} 16 & 6 \\ 12 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 18 \\ 24 & -6 \end{bmatrix}$

(7.2)

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ $A - \lambda E_3$ singulier? $\rightarrow \det(A - \lambda E_3) = 0$

$$A - \lambda E_3 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix} \rightarrow \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (4-\lambda)(\lambda^2 - 1)$$

$\Leftrightarrow (4-\lambda)(\lambda^2-1) = 0$

$\Leftrightarrow 4 - \lambda = 0$ of $\lambda^2 - 1 = 0$

$\Leftrightarrow \lambda = 4$ of $\lambda = 1$ of $\lambda = -1$

7.3

1. $$A.B = \underbrace{\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}}_{3\times 2} \cdot \underbrace{\begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 2.2+8.3 & 2.0+8.8 \\ 3.2+0.3 & 3.0+0.8 \\ 5.2+1.3 & 5.0+1.8 \end{bmatrix} = \begin{bmatrix} 28 & 64 \\ 6 & 0 \\ 13 & 8 \end{bmatrix}$$

2. $$B.A = \underbrace{\begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}}_{3\times 2} = /$$

3. $$B.C = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2.7+0.6 & 2.2+0.3 \\ 3.7+8.6 & 3.2+8.3 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 69 & 30 \end{bmatrix}$$

4. $$C.B = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 7.2+2.3 & 7.0+2.8 \\ 6.2+3.3 & 6.0+3.8 \end{bmatrix} = \begin{bmatrix} 20 & 16 \\ 21 & 24 \end{bmatrix}$$

5. $$A.C = \underbrace{\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 2.7+8.6 & 2.2+8.3 \\ 3.7+0.6 & 3.2+0.3 \\ 5.7+1.6 & 5.2+1.3 \end{bmatrix} = \begin{bmatrix} 62 & 28 \\ 21 & 6 \\ 41 & 13 \end{bmatrix}$$

6. $$C.A = \underbrace{\begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}}_{3\times 2} = /$$

7. $$B.D = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.2+0.3 \\ 3.2+8.3 \end{bmatrix} = \begin{bmatrix} 4 \\ 30 \end{bmatrix}$$

8. $$D.B = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} \underbrace{\begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}}_{2\times 2} = /$$

(9) $A.D = \underbrace{\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}}_{3\times 2} \cdot \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} = \begin{bmatrix} 2.2+8.3 \\ 8.2+0.3 \\ 5.2+1.3 \end{bmatrix} = \begin{bmatrix} 28 \\ 6 \\ 13 \end{bmatrix}$

(10) $D.A = \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} \cdot \underbrace{\begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix}}_{3\times 2} = /$

(7.4) (1) $A+B^T = \begin{bmatrix} -2 & 4 & 7 \\ 3 & -1 & 8 \\ 2 & 1 & -3 \end{bmatrix} + \begin{bmatrix} 9 & -1 & 0 \\ 2 & -1 & 2 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 7 \\ 5 & -2 & 10 \\ 4 & 5 & 0 \end{bmatrix}$

(2) $A.B = \begin{bmatrix} -2 & 4 & 7 \\ 3 & -1 & 8 \\ 2 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 9 & 2 & 2 \\ -1 & -1 & 4 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -29-41 & -22-4+14 & -4+16+21 \\ 27+11+0 & 6+1+16 & 6-4+24 \\ 18-1+0 & 4-1-6 & 4+4-9 \end{bmatrix}$

$= \begin{bmatrix} -22 & 6 & 33 \\ 28 & 23 & 26 \\ 17 & -3 & -1 \end{bmatrix}$

(3) $B.C^T = \underbrace{\begin{bmatrix} 9 & 2 & 2 \\ -1 & -1 & 4 \\ 0 & 2 & 3 \end{bmatrix}}_{3\times 3} \cdot \underbrace{\begin{bmatrix} 7 & -2 & 3 \\ -4 & 3 & 1 \end{bmatrix}}_{2\times 3} = /$

(4) $A.C = \underbrace{\begin{bmatrix} -2 & 4 & 7 \\ 3 & -1 & 8 \\ 2 & 1 & -3 \end{bmatrix}}_{3\times 3} \cdot \underbrace{\begin{bmatrix} 7 & -4 \\ -2 & 3 \\ 3 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} -14-8+21 & 8+12+7 \\ 21+2+24 & -12-3+8 \\ 14-2-9 & -8+3-3 \end{bmatrix}$

$= \begin{bmatrix} -16 & 27 \\ 47 & -7 \\ 3 & -8 \end{bmatrix}$

7.5 ① $\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 4 \\ 2 & 3 & 0 \end{bmatrix}}_{3\times 3} \cdot \underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 0 & 1 \\ 24+12 & 20 \\ 16 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 36 & 20 \\ 16 & 3 \end{bmatrix}$

② $\underbrace{\begin{bmatrix} 6 & 5 & 1 \\ 3 & 0 & 4 \end{bmatrix}}_{2\times 3} \cdot \underbrace{\begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 24+25 & -6+10+1 \\ 12 & -3+4 \end{bmatrix} = \begin{bmatrix} 49 & 5 \\ 12 & 1 \end{bmatrix}$

③ $\begin{bmatrix} 3 & 2 & 0 \\ 4 & 2 & -7 \end{bmatrix} \cdot \begin{bmatrix} x & y & z \end{bmatrix}^T = \underbrace{\begin{bmatrix} 3 & 2 & 0 \\ 4 & 2 & -7 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1}$

$= \begin{bmatrix} 3x+2y \\ 4x+2y-7z \end{bmatrix}$

④ $\underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{1\times 3} \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 7a+c & 2b+4c \end{bmatrix}$

7.6 $A = \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix}$ en $A.A = A$

$\rightarrow \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix} \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix} = \begin{bmatrix} x^2 & x+y \\ 0 & y^2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} x^2 & x+y \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ 0 & y \end{bmatrix}$

$\Leftrightarrow \begin{cases} x^2 = x \\ x+y = 1 \\ y^2 = y \end{cases} \Leftrightarrow \begin{cases} x^2 - x = 0 \\ x+y = 1 \\ y^2 - y = 0 \end{cases} \Leftrightarrow \begin{cases} x(x-1) = 0 \\ x+y = 1 \\ y(y-1) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 1 \end{cases}$ of $\begin{cases} x = 1 \\ y = 0 \end{cases}$

(7.7) $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ wanneer $A \cdot A = A$

<u>voor $n=2$</u>

$$\begin{bmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

$$\begin{cases} a_{11}^2 = a_{11} \\ a_{22}^2 = a_{22} \end{cases} \Leftrightarrow \begin{cases} a_{11} = 0 \text{ of } a_{11} = 1 \\ a_{22} = 0 \text{ of } a_{22} = 1 \end{cases}$$

dus $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

<u>voor $n=3$</u>

$$\begin{cases} a_{11}^2 = a_{11} \\ a_{22}^2 = a_{22} \\ a_{33}^2 = a_{33} \end{cases} \Leftrightarrow \begin{cases} a_{11} = 0 \text{ of } a_{11} = 1 & \rightarrow 2 \\ a_{22} = 0 \text{ of } a_{22} = 1 & \rightarrow 2 \\ a_{33} = 0 \text{ of } a_{33} = 1 & \rightarrow \underline{2} \end{cases}$$

8 mogelijkheden

Veralgemening: voor matrix van orde n
$\rightarrow 2^n$

(7.8) (1) $\text{adj}(A) = \begin{bmatrix} -2 & -8 & -12 \\ -2 & 19 & -12 \\ 10 & -14 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & 10 \\ -8 & 19 & -14 \\ -12 & -12 & 6 \end{bmatrix}$

$$A \cdot \text{adj}A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -2 & 2 \\ -6 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -2 & 10 \\ -8 & 19 & -14 \\ -12 & -12 & 6 \end{bmatrix} = \begin{bmatrix} -54 & 0 & 0 \\ 0 & -54 & 0 \\ 0 & 0 & -54 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ -4 & -2 & 2 \\ -6 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 19 & 2 & 3 \\ 8 & -2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 19 & 2 \\ 8 & -2 \end{vmatrix} = -38 - 16 = -54$$

② $\operatorname{adj} A = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

$A.\operatorname{adj} A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -7+2+3 & 6-6 & -1-2+3 \\ -7+3+4 & 6-8 & -1-3+4 \\ -7+4+3 & 6-6 & -1-4+3 \end{bmatrix}$

$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$

③ $\operatorname{adj} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$A.\operatorname{adj} A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

④ $\operatorname{adj} A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -5 & -2 \\ -2 & 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$

$A.\operatorname{adj} A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 4-4 & -2+2 \\ 2-2 & 8-5 & -4+4 \\ 3-4+1 & 12-10-2 & -6+8+1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 3 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -4 \\ 2 & -5 \end{vmatrix} = -5+8 = 3$$

(5) $$\operatorname{adj} A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$A \operatorname{adj} A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & -3 & -5 \end{vmatrix} = 5-12 = -7$$

(6) $$\operatorname{adj} A = \begin{bmatrix} 2 & 2 & 1 & -2 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -4 & 0 & 0 & 10 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ -2 & 0 & 0 & 10 \end{bmatrix}$$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ -2 & 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 2(5-2) = 6$$

7.9

1. $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

$$\operatorname{adj} A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix}$$

$$= 6 - 2.2 + 3(-3) = -7$$

$$A^{-1} = \frac{1}{-7} \cdot \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -6/7 & -1/7 & 5/7 \\ 2/7 & 5/7 & -4/7 \\ 3/7 & -3/7 & 1/7 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\operatorname{adj} A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -5 & -2 \\ -2 & 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 + 2(4-3) = 3$$

$$A^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 4/3 & -2/3 \\ -2/3 & -5/3 & 4/3 \\ 1/3 & -2/3 & 1/3 \end{bmatrix}$$

3. $$\operatorname{adj} A = \begin{bmatrix} 30 & -15 & 0 \\ 0 & 0 & 0 \\ -10 & 5 & 0 \end{bmatrix}^T = \begin{bmatrix} 30 & 0 & -10 \\ -15 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 6 & 1 \\ 6 & 9 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 9 \end{vmatrix} + 3 \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 30 - 2.15 + 3.0 = 0$$

$\rightarrow A^{-1}$ bestaat niet

(4) $$\left[\begin{array}{cccc|cccc} 5 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{5R_2-R_1 \\ 5R_4-R_1}}{=} \left[\begin{array}{cccc|cccc} 5 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 8 & -1 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 0 & 0 & 5 \end{array}\right]$$

$$\underset{\substack{3R_1-2R_4 \\ 3R_2-8R_4 \\ 3R_3-R_4}}{=} \left[\begin{array}{cccc|cccc} 15 & 0 & 0 & 0 & 5 & 0 & 0 & -10 \\ 0 & 15 & 0 & 0 & 5 & 15 & 0 & -40 \\ 0 & 0 & 6 & 0 & 1 & 0 & 3 & -5 \\ 0 & 0 & 0 & 3 & -1 & 0 & 0 & 5 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1 & 0 & -8/3 \\ 0 & 0 & 1 & 0 & 1/6 & 0 & 1/2 & -5/6 \\ 0 & 0 & 0 & 1 & -1/3 & 0 & 0 & 5/3 \end{array}\right]$$

(7.10) (1) $\operatorname{adj} A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

$\det A = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = 6 - 2 \cdot 2 + 3 \cdot (-3) = -7$

$A^{-1} = \begin{bmatrix} -6/7 & -1/7 & 5/7 \\ 2/7 & 5/7 & -4/7 \\ 3/7 & -3/7 & 1/7 \end{bmatrix}$

$X = A^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 10/7 & -6/7 - 3/7 + 30/7 \\ -8/7 & 2/7 + 15/7 - 24/7 \\ 2/7 & 3/7 - 9/7 + 6/7 \end{bmatrix} = \begin{bmatrix} 10/7 & 3 \\ -8/7 & -1 \\ 2/7 & 0 \end{bmatrix}$

(2) $\operatorname{adj} A \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

$\det A = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -7 + 2 + 3 = -2$

$$A^{-1} = \begin{bmatrix} 7/2 & -3 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 7/2 & -3 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix} = \begin{bmatrix} -3/2 & 3 & -3/2 \\ -3/2 & 2 & -1/2 \\ 3/2 & -3 & 5/2 \end{bmatrix}$$

③ $\operatorname{adj} A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$\det A = -2$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 5/2 & 5/2 \end{bmatrix}$$

④ $\left[\begin{array}{cccc|cccc} 5 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ (uitwerking van inverse matrix, zie vorige oefening)

$$\Rightarrow A^{-1} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1 & 0 & -8/3 \\ 0 & 0 & 1 & 0 & 1/6 & 0 & 1/2 & -5/6 \\ 0 & 0 & 0 & 1 & -1/3 & 0 & 0 & 5/3 \end{array}\right]$$

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1/6 & 2 & 3/2 & -11/6 \\ 1/6 & 1 & -1/2 & -11/6 \end{bmatrix}$$

7.11 zelfde methode als 7.12, maar meer werk (volgt later)

7.12

	S_1	S_2	F_c	TO
S_1	40	25	35	100
S_2	60	50	15	125

(1) $T = \begin{bmatrix} 2/5 & 1/5 \\ 3/5 & 2/5 \end{bmatrix} \quad t_{ij} = \frac{x_{ij}}{x_j}$

$X = TX + F \Leftrightarrow \underbrace{(E_n - T)}_{L}X = F$

$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/5 & 1/5 \\ 3/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -3/5 & 3/5 \end{bmatrix}$

$\det L = \begin{vmatrix} 3/5 & -1/5 \\ -3/5 & 3/5 \end{vmatrix} = \frac{9}{25} - \frac{3}{25} = \frac{6}{25}$

$\operatorname{adj} L = \begin{bmatrix} 3/5 & 3/5 \\ 1/5 & 3/5 \end{bmatrix}^T = \begin{bmatrix} 3/5 & 1/5 \\ 3/5 & 3/5 \end{bmatrix}$

$L^{-1} = \frac{25}{6} \begin{bmatrix} 3/5 & 1/5 \\ 3/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 5/2 & 5/6 \\ 5/2 & 5/2 \end{bmatrix}$

$\hookrightarrow m_1 = 5 \searrow m_2 = \frac{10}{3}$

(2) $F = LX \qquad \Delta F = L\,\Delta X$

$X = L^{-1}F \qquad \Delta X = L^{-1}\Delta F$

<u>Geg</u>: $\Delta F_1 = -5$

$\Delta F_2 = 0$

<u>Gevr</u>: ΔX_2

<u>Opl</u>: $\Delta X = L^{-1}\Delta F$

$\rightarrow \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} = \begin{bmatrix} 5/2 & 5/6 \\ 5/2 & 5/2 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -25/2 \\ -25/2 \end{bmatrix}$

③ Geg: $\Delta F_1 = -2$

$\Delta F_2 = 2$

$\rightarrow \Delta X = C^{-1} \cdot \Delta F$

$$= \begin{bmatrix} 5/2 & 5/6 \\ 5/2 & 5/2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -10/3 \\ 0 \end{bmatrix}$$

④ $\Delta X_1 = 2 \quad F = C.X$

$\Delta X_2 = 1$

$$\begin{bmatrix} 3/5 & -1/5 \\ 3/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3/5 \end{bmatrix}$$

⑤ $\Delta X = 4$

$F_2 =$ constante

Geg: $\Delta X_1 + \Delta X_2 = 4$

$\Delta F_2 = 0$

Gevr: $\Delta F_1 = x$

Opl: $\Delta X = C^{-1} \cdot \Delta F$

$$\Leftrightarrow \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} = C^{-1} \cdot \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 4 = \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$\Leftrightarrow 4 = \begin{bmatrix} 5 & 10/3 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}$$

$$= 5x + 0$$

$$\Leftrightarrow x = \frac{4}{5}$$

7.13 (volgt later)