limieten

continuiteit: i/e punt
i/e open gebied
i/e gesloten gebied

Stelling van Weierstraß

Partiële afgeleiden van eerste orde

Stelling van Schwarz en Young

Lemma

Kettingregel

Eigenschappen directionele afgeleiden

Differentiaal v/e functie van $\mathbb{R}$ naar $\mathbb{R}$

De differentiaal df als benadering v/d toename Δf

Impliciete functiestelling ($n=1$ & $n>1$)

Homogene functies (Def + eigenschap)
→ partiële afgeleiden

Stelling van Euler

Wet v/d afnemende meeropbrengsten

Def. kwadratische vorm + matrix

$x^TAx = x^TBx$

a (lokaal) extremum → partiële afgeleide 1ste orde $=0$

stationair of kritisch punt

Bew verband extremum & $D^2f(a,b)$

voorraadbeheer: discontinu

Lagrangemethode

Differentie v/e functie

methode v/d onbepaalde coëfficiënten

Wiskunde

Inleidende begrippen

1 1) $D = 25 - 24 = 1$ $\quad x_1 = \frac{5+1}{2} = 3$

$x_2 = \frac{5-1}{2} = 2$

2)

	1	-2	1
1		1	-1
	1	-1	$\boxed{0}$

$(x-1)^2 = 0 \quad x = 1$

3) $D < 0 \Rightarrow$ geen oplossing

2 1) $D = m^2 - 64$

Geval 1: $D > 0$: $m^2 - 64 > 0$

$\Leftrightarrow m > 8 \vee m > -8$

$x = \frac{-m \pm \sqrt{m^2 - 64}}{2}$

Geval 2: $D = 0$: $m^2 - 64 = 0$

$\Leftrightarrow m = 8 \vee m = -8$

$x = \frac{-m}{2}$

$\Leftrightarrow x = 4 \vee x = -4$

Geval 3: $D < 0$: $m^2 - 64 < 0$

$\rightarrow$ geen oplossing

2) $D = 4 - 4m^2 = 4(1 - m^2)$

Geval 1 $D > 0$ $\quad 4(1-m^2) > 0$

$\Leftrightarrow$

	-1		1	
-	0	+	0	-

$-1 < m < 1$

$x = \frac{-2 \pm 2\sqrt{1-m^2}}{2m} = \frac{-1 \pm \sqrt{1-m^2}}{m}$

Geval 2 $D = 0$ $\quad m = 1 \vee m = -1$

$x = \frac{-1}{m} \Leftrightarrow x = -1 \vee x = 1$

Geval 3 $D<0 \rightarrow$ geen oplossingen

Geval 4 $m=0 \Rightarrow x=0$

3) $D = 16 + 4m = 4(4+m)$

Geval 1 $D>0$ $\quad 4(4+m)>0$

$\Leftrightarrow m>-4$

$$x = \frac{4 \pm 2\sqrt{4+m}}{-2} = \frac{2 \pm \sqrt{4+m}}{-1}$$

Geval 2 $D=0$ $\quad m=-4$

$x=-2$

Geval 3 $D<0 \rightarrow$ geen opl.
$m<-4$

4) $D = m^2 - 4m = m(m-4)$

Geval 1 $m=0 \Rightarrow 1 \neq 0 \rightarrow$ niet mogelijk

Geval 2 $D>0$

		0		4	
	+	0	−	0	+

$m<0 \vee m>4$

$$x = \frac{-m \pm \sqrt{m^2-4m}}{2m}$$

Geval 3 $D=0 \Leftrightarrow \cancel{m=0} \vee m=4$
strijdig

$$x = \frac{-m}{2m} = \frac{-1}{2}$$

Geval 4 $D<0 \rightarrow$ opl. $\varnothing$

3 1)

	4	21	21	4
-1		-4	-17	-4
	4	17	4	$\boxed{0}$
-4		-16	-4	
	4	1	$\boxed{0}$	

$(x+1)(x+4)(4x+1)=0$

$\Leftrightarrow x=-1 \vee x=-4 \vee x=-1/4$

2)

	3	-8	8	-3
1		3	-5	3
	3	-5	3	$\boxed{0}$

$(x-1)(3x^2-5x+3)=0$

$\Leftrightarrow x=1$

1

3)

60	-44	-25	-44	60
60				

$x^2(60x^2 - 44x - 25 - \frac{44}{x} + \frac{60}{x^2})$

Stel $y = x + \frac{1}{x}$ $\qquad 60y^2 - 44y - 145 = 0$

$y > 0$

4)

	1	-6	-2	40
+4		4	-8	-40
	1	-2	-10	$\boxed{0}$

$(x-4)(x^2 - 2x - 10) = 0$

$D = 4 + 40 = 44$

$x = \frac{2 \pm \sqrt{44}}{2}$

$\Leftrightarrow x = 4 \vee x = \frac{2 \pm \sqrt{44}}{2} = 1 \pm \sqrt{11}$

5)

	1	2	-23	-60
5		5	35	60
	1	7	12	$\boxed{0}$

$(x-5)(x^2 + 7x + 12) = 0$

$D = 49 - 60 < 0$

$\Leftrightarrow x = 5$

6)

	1	0	-2	-3	-2
2		2	4	4	2
	1	2	2	1	$\boxed{0}$
-1		-1	-1	-1	
	1	1	1	0	

$(x-2)(x+1)(x^2 + x + 1) = 0$

$D = 1 - 4 = -3$

$\Leftrightarrow x = 2 \vee x = -1$

7) $[(x+1)(x-1)^2]^2$

$x = -1 \vee x = 1$

8) $x = -1/2 \vee x = 2/3 \vee x = 5/2$

4 i) $\begin{cases} 2x\,(1 - (x^2 + y^2)) = 0 \\ 2y\,(1 - (x^2 + y^2)) = 0 \end{cases}$ VW $x^2 + y^2 \neq 0$

$\Leftrightarrow \begin{cases} x = 0 \vee x^2 + y^2 = 1 \\ y = 0 \vee x^2 + y^2 = 1 \end{cases}$ $(x, y) \neq (0,0)$

$\Leftrightarrow \begin{cases} y = 0 \\ x = 1 \end{cases} \vee \begin{cases} y = 0 \\ x = -1 \end{cases}$

2) $\begin{cases} y^2 x - 2y = 0 \\ 2xy - 2\ln x = 0 \end{cases}$ VW $x \neq 0$ $\Leftrightarrow \begin{cases} y(yx - 2) = 0 \\ 2xy - 2\ln x = 0 \end{cases} \Leftrightarrow \begin{cases} x = e^{2xy/2} \end{cases}$

$\begin{cases} y = 0 \\ x = 1 \end{cases} \vee \begin{cases} y = 2/x \\ x = e^2 \end{cases}$

3) stel $(y+x^2)=p$

$$\begin{cases} 2xe^{-p} + p(-2x)e^{-p} = 0 \\ e^{-p} - pe^{-p} = 0 \end{cases} \Leftrightarrow \begin{cases} 2xe^{-p}(1+p) = 0 \\ e^{-p}(1-p) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} p = -1 \\ p = 1 \end{cases} \Leftrightarrow \begin{cases} y + x^2 = -1 \\ y + x^2 = 1 \end{cases} \Leftrightarrow$$

4) $$\begin{cases} 2x(1-\lambda) + 6(1-\lambda) = 0 \\ -6y - 10 + \lambda = 0 \\ y + 6 - (x+3)^2 = 0 \end{cases} \Leftrightarrow \begin{cases} (1-\lambda)(2x+6) = 0 \\ \ldots \\ \ldots \end{cases}$$

$$\begin{cases} \lambda = 1 \\ y = -3/2 \\ x = -3 + \frac{3\sqrt{2}}{2} \end{cases} \vee \begin{cases} x = -3 \\ y = -6 \\ \lambda = 26 \end{cases} \vee \begin{cases} \lambda = 1 \\ y = -3/2 \\ x = -3 - \frac{3\sqrt{2}}{2} \end{cases}$$

5 $u + v = (1, 3)$ $3z = (0, 3, 3)$ $u - v = (1, 1)$

$-4w = (-4, 12)$ $2v = (0, 2)$ $3x + 2z = (3, 8, 2)$

$u + z = /$ $u + 2v = (1, 4)$ $-2x = (-2, -4, 0)$

$1/2\, w + 2x = /$

6 1) $\cancel{\sqrt{(3-0)^2 + (4+3)^2} = \sqrt{9+144} = \sqrt{153}}$

$\sqrt{3^2 + 4^2} = 5$ $\quad |(0, -3)| = \sqrt{0^2 + (-3)^2} = 3$

2) $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \quad \wedge \quad \sqrt{4 + 16 + 9} = \sqrt{29}$

3) $\sqrt{1 + 4 + 9 + 16} = \sqrt{30} \quad \wedge \quad \sqrt{9 + 9} = \sqrt{18} = 3\sqrt{2}$

7 1) $\sqrt{(0-3)^2 + (0+4)^2} = \sqrt{25} = 5$

2) $\sqrt{(-1)^2 + (2)^2 + (-6)^2} = \sqrt{1 + 4 + 36} = \sqrt{41}$

3) $\sqrt{0^2 + 2^2 + 4^2 + 4^2} = \sqrt{4 + 16 + 16} = 6$

8 1) rico $= (5-3, 0-0) = (2, 0)$

$$\begin{cases} x = 3 + 2t \\ y = 0 + 0t \end{cases} \quad \text{of} \quad \begin{cases} x = 3s + 5t \\ y = 0s + 0t \end{cases}$$

2) rico $= (0-1, 1-0) = (-1, 1)$

$$\begin{cases} x = 1 - t \\ y = t \end{cases} \quad \text{of} \quad \begin{cases} x = s \\ y = t \end{cases}$$

3) rico: $(1,1,-1)$

$$\begin{cases} x = 1 + \cancel{F} t \\ y = 0 + t \\ z = 1 \cancel{+} - t \end{cases}$$

4) rico: $(2,4)$

$$\begin{cases} x = 1 + 2t \\ y = 2 + 4t \end{cases} \quad \text{of} \quad \begin{cases} x = s + 3t \\ y = 2s + 6t \end{cases}$$

5) rico: $(3,9) \sim \boxed{(1,3)}$

$$\begin{cases} x = 1 + 3t \\ y = 1 + 9t \end{cases}$$

10 1) $\begin{cases} 11 = 1 + 5t \\ 14 = 2 + 6t \\ 17 = 3 + 7t \\ 20 = 4 + 8t \end{cases} \Leftrightarrow t = 2 \Rightarrow$ op rechte

2) $\begin{cases} 4 = 1 + t + \cancel{s} \\ 3 = 2 + t + s \\ 1 = 3 + s \end{cases} \quad \begin{cases} t = 3 \\ s = -2 \\ t + s = 1 \end{cases} \Rightarrow$ op rechte

3) $\begin{cases} t = 0 \\ s = 0 \\ 0 = 1 \end{cases} \Rightarrow$ strijdig : niet op rechte

Functies van meerdere veranderlijken

1 1) $f(x,y) = xy\ln(xy)$

- $\frac{\partial f}{\partial x} = y\ln(xy) + xy\frac{1}{xy}y$

$= y(\ln(xy)+1)$

- $\frac{\partial f}{\partial y} = x\ln xy + xy\frac{1}{xy}x$

$= x(\ln xy + 1)$

2) $f(u,v,w) = e^{au} + bv^2 + cw^3$

- $\frac{\partial f}{\partial u} = ae^{au} + bv^2 + cw^3$

- $\frac{\partial f}{\partial v} = e^{au} + 2bv + cw^3$

- $\frac{\partial f}{\partial w} = e^{au} + bv^2 + 3cw^2$

3) $f(x,y) = \sqrt{xy}$

- $\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{xy}}y$

- $\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{xy}}x$

4) $f(x,y) = ye^{xy}$

- $\frac{\partial f}{\partial x} = y^2e^{xy}$
- $\frac{\partial f}{\partial y} = e^{xy} + ye^{xy}x = e^{xy}(1+xy)$

5) $f(x,y,z) = \frac{x^2y^2z^2}{x+y+z}$

- $\frac{\partial f}{\partial x} = \frac{(x+y+z)2xy^2z^2 - (x^2y^2z^2)(y+z)}{(x+y+z)^2}$

$= \frac{xy^2z^2(2x+2y+2z-xy-xz)}{(x+y+z)^2}$

6) $f(x,y) = \ln(\sin^2 x + \cos^2 y)$

- $$\frac{\partial f}{\partial x} = \frac{1}{\sin^2 x + \cos^2 y} 2\sin x \cos x = \frac{\sin 2x}{\sin^2 x + \cos^2 y}$$

- $$\frac{\partial f}{\partial y} = \frac{1}{\sin^2 x + \cos^2 y}(-2\cos y \sin y) = \frac{-\sin 2x}{\sin^2 x + \cos^2 y}$$

2 1) $f(x,y) = e^{-x}\sin(x+y)$

- $$\frac{\partial f}{\partial x} = -1e^{-x}\sin(x+y) + e^{-x}\cos(x+y)$$

  $$\to (0,\tfrac{\pi}{4}): -\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$$

- $$\frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$$

  $$\to (0,\tfrac{\pi}{4}): \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

2) $f(x,y) = \arctan\left(\left(\frac{x+y}{x-y}\right)^2\right)$

- $$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^4} \cdot 2\left(\frac{x+y}{x-y}\right)\left(\frac{(\cancel{x}-y)-(\cancel{x}+y)}{(x-y)^2}\right)$$

  $$= \frac{-4y+2x}{\left(1-\left(\frac{x+y}{x-y}\right)^4\right)(x-y)^3} = \frac{-4y+2x}{\frac{(x-y)^4-(x+y)^4}{(x-y)}}$$

  $$= \frac{(x+y)(x-y)(-4y)}{(x-y)^4+(x+y)^4} = \frac{-4y(x^2-y^2)}{(x-y)^4+(x+y)^4}$$

  $$\to (2,1): \frac{-4(4-1)}{1+81} = \frac{-12}{82} = \frac{-6}{41}$$

- $$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^4} 2\left(\frac{x+y}{x-y}\right)\frac{(x-y)+(x+y)}{(x-y)^4}$$

  $$= \frac{4x(x^2-y^2)}{(x-y)^4+(x+y)^4} \to (2,1): \frac{8 \cdot 3}{82} = \frac{24}{82} = \frac{12}{41}$$

1) 3 $\ln\sqrt{x^2+y^2} = \frac{1}{2}\ln x^2+y^2$

$$\frac{1}{2}\ln(x^2+y^2) = \arctan\frac{y}{x}$$

- $$\Leftrightarrow \frac{1}{2}\frac{1}{x^2+y^2}2x\,dx + \frac{1}{2}\frac{1}{x^2+y^2}2y\,dy = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)dx + \ldots \frac{1}{x}dy$$

$\Leftrightarrow \frac{x+y}{x^2+y^2}dx + \frac{y-x}{x^2+y^2}dy = 0$

$$\frac{-\frac{y}{x^2}\cdot x^2}{\left(1+\frac{y^2}{x^2}\right)\cdot x^2}dx + \frac{\frac{1}{x}\cdot x^2}{\left(1+\frac{y^2}{x^2}\right)x^2}dy$$

$\Leftrightarrow \frac{dy}{dx} = \frac{x+y}{x-y} = y'$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

• ~~$y'' = \frac{(x-y)-(x+y)}{(x-y)^2}dx + \frac{(x-y)+(x+y)}{(x-y)^2}dy = 0$~~

~~$\Leftrightarrow \frac{(x^2-y^2)dx + (x^2-y^2)dy}{(x-y)^2} = 0$~~

$y'' = \frac{(x-y)-(x+y)}{(x-y)^2}dx + \frac{(x-y)+(x+y)}{(x-y)^2}dy = 0$

$\Leftrightarrow \frac{-2y\,dx}{(x-y)^2} + \frac{2x\,dy}{(x-y)^2} = 0$

$\Leftrightarrow$

$\frac{\partial f}{\partial x} = \frac{x+y}{x^2+y^2}dx \qquad \frac{\partial f}{\partial y} = \frac{y-x}{x^2+y^2}dy$

$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)-(x+y)2x}{(x^2+y^2)^2}dx \qquad \frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)-(y-x)2y}{(x^2+y^2)^2}dy$

$= \frac{x^2+y^2-2x^2-2xy}{(x^2+y^2)^2}dx \qquad = \frac{x^2+y^2-2y^2+2xy}{(x^2+y^2)^2}$

$= \frac{-x^2-2xy+y^2}{(x^2+y^2)^2} \qquad = \frac{x^2+2xy-y^2}{(x^2+y^2)^2}$

$y'' = - \frac{-(x^2+2xy-y^2)}{}$

4 1) $f(x,y,z) = x^3 + y^3 + z^3$ met $x = e^t \cos t$, $y = e^t \sin t$ $\wedge$ $z = e^t$

- $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= 3x^2 \cdot (e^t \cos t - e^t \sin t) + 3y^2 (e^t \sin t + e^t \cos t) + 3z^2 e^t$$

$$= 3x^2 e^t (\cos t - \sin t) + 3y^2 e^t (\sin t + \cos t) + 3z^2 e^t$$

$$= 3(e^t \cos t)^2 e^t (\cos t - \sin t) + 3(e^t \sin t)^2 (\sin t + \cos t) + 3e^{2t} e^t$$

$$\cancel{= 3e^{3t}(\cos^3 t - \sin t \cos^2 t + \sin^3 t + \sin^2 t \cos t + 1)}$$

$$= 3e^{3t}\cos^3(t) - 3e^{3t}\cos^2 t \sin t + 3e^{3t}\sin^3 t + 3e^{3t}\cos t \sin^2(t) + 3e^{3t}$$

$$= 3e^{3t}(\cos^3 t - \cos^2 t \sin t + \sin^3 t + \cos t \sin^2 t + 1)$$

$\overset{?}{=} \ldots$

2) $f(x,y,z) = \frac{x}{y} + \frac{y}{z}$ met $x = 2t$, $y = \frac{1}{t}$ $\wedge$ $z = t^2$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{1}{y} \cdot 2 + \left(\frac{-x}{y^2} + \frac{1}{z}\right)\frac{-1}{t^2} + \frac{-y}{z^2} \cdot 2t$$

$$= \frac{2}{y} + \frac{x}{y^2 t^2} - \frac{1}{z t^2} - \frac{2yt}{z^2}$$

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$$= 2t + 2t - \frac{1}{t^4} - \frac{2}{t^4}$$

$$= \frac{-3}{t^4} + 4t$$

3) $f(x,y,z) = \ln\frac{x}{y} + \ln\frac{x}{z} + \ln\frac{y}{z}$ $x = t\cos t$, $y = t\sin t$ $z = t\tan t$

$$\frac{\partial f}{\partial t} = \left(\frac{y}{x}\cdot\frac{1}{y} + \frac{z}{x}\cdot\frac{1}{z}\right)(\cos t - t\sin t) + \left(\frac{y}{x}\cdot\frac{-x}{y^2} + \frac{z}{y}\cdot\frac{1}{z}\right)(\sin t + t\cos t)$$

$$+ \left(\frac{z}{x}\cdot\frac{-x}{z^2} + \frac{z}{y}\cdot\frac{-y}{z^2}\right)\left(\tan t + \frac{t}{\cos^2 t}\right)$$

$$= \left(\frac{2}{t\cos t}\right)(\cos t - t\sin t) + \left(\frac{2}{t\sin t}\right)(\sin t + t\cos t) + \left(\frac{-2}{t\tan t}\right)\left(\tan t + \frac{t}{\cos^2 t}\right)$$

$$= \frac{2}{t}\left(1 - t\frac{\sin t}{\cos t} \cancel{+1} + t\frac{\cos t}{\sin t} \cancel{-1} - \frac{1}{\tan t \cos^2 t}\right)$$

$$= \frac{2}{t}\left(\frac{\sin t\cos t + \sin t + \cos t - 1}{\sin t \cos t}\right)$$

$= \ldots$

5 i) $\begin{cases} x = 1+4t \\ y = 1-3t \end{cases}$ $\quad F(t) = ((1+4t),(1-3t))$

$F'(t) = 2x\cdot 4 + 2y\cdot(-3) = 8x - 6y$

$F'(0) = 2 = \dfrac{\partial f}{\partial v}(1,1)$

2) $\begin{cases} x = 1+t \\ y = 1+t \end{cases}$ $\quad F(t) = ((1+t),(1+t))$

$$F'(t) = \frac{1}{\sqrt{x^2+y^2}}\,\frac{1}{2}\,\frac{1}{\sqrt{x^2+y^2}}\cdot 2x + \frac{1}{\sqrt{}}\,\frac{1}{2}\,\frac{1}{\sqrt{}}\cdot 2y$$

$$= \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} = \frac{x+y}{x^2+y^2}$$

$\dfrac{\partial f}{\partial v}(1,1) = \dfrac{dF}{dt}(0) = 1$

3) $\begin{cases} x = 2+2t \\ y = 2+t \end{cases}$ $\quad F(t) = ((2+2t),(2+t))$

$$\frac{\partial f}{\partial v} = \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 2.\left(y^2+2xy\right) + (2xy+x^2)$$

$\Rightarrow (2,2):\ 2(4+8) + (8+4) = 36$

7 i) $f(x,y,z) = \ln(x^2+y^2+2z)$ met $x = t+s$, $y = t-s$, $z = 2ts$

$$\frac{dF}{dt} = \underset{a}{\frac{\partial f}{\partial t}} + \underset{b}{\frac{\partial f}{\partial s}} = \underset{1}{\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}} + \underset{2}{\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}} + \underset{3}{\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}} + \underset{4}{\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}} + \underset{5}{\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}} + \underset{6}{\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}}$$

1 $\quad \dfrac{2x}{x^2+y^2+2z}\cdot 1 = \dfrac{2(t+s)}{(t+s)^2+(t-s)^2+4ts} = \dfrac{t+s}{(t+s)^2} = \dfrac{1}{t+s}$

2 $\quad \dfrac{2y}{x^2+y^2+2z}\cdot 1 = \dfrac{2(t-s)}{\dots} = \dfrac{t-s}{(t+s)^2}$

3 $\quad \dfrac{2}{x^2+y^2+2z}\cdot 2s = \dfrac{2s}{(t+s)^2}$

a $\quad \dfrac{2}{t+s}$

4 $\quad \dfrac{2x}{x^2+y^2+2z}\cdot 1 = \dfrac{1}{t+s}$ $\qquad$ 5 $\quad \dfrac{2y}{x^2+y^2+2z}\cdot(-1) = \dfrac{-(t-s)}{(t+s)^2}$

6 $\quad \dfrac{2}{x^2+y^2+2z}\cdot 2t = \dfrac{2t}{(t+s)^2}$ $\qquad$ b $\quad \dfrac{2}{t+s}$

$$\left(\Rightarrow \frac{dF}{dt} = \frac{2}{t+s} + \frac{2}{t+s} = \frac{4}{t+s}\right)$$

4 1) $f(x,y,z) = x^3 + y^3 + z^3$ met $x = e^t \cos t$, $y = e^t \sin t$ en $z = e^t$

$$\frac{dF}{dt}(t) = \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt}}_{(1)} + \underbrace{\frac{\partial f}{\partial y}\frac{dy}{dt}}_{(2)} + \underbrace{\frac{\partial f}{\partial z}\frac{dz}{dt}}_{(3)}$$

(1) $3x^2 \frac{dx}{dt} = 3(e^{2t}\cos^2 t) . (-e^t \sin t + e^t \cos t)$

(2) $3y^2 \frac{dy}{dt} = 3(e^{2t}\sin^2 t) . (e^t \cos t + e^t \sin t)$

(3) $3z^2 \frac{dz}{dt} = 3e^{2t} . e^t$

(1) + (2) + (3) $= 3e^{3t}(-\sin t \cos^2 t + \cos^2 t \cos t + \sin^2 t \cos t + \sin^2 t \sin t + 1)$

$= 3e^{3t}(\cos t(\cos^2 t + \sin^2 t) + \sin t(\sin^2 t - \cos^2 t) + 1$

$= 3e^{3t}(\cos t + \sin t \cos 2t + 1)$

2) $f(x,y,z) = \ln\frac{x}{y} + \ln\frac{x}{z} + \ln\frac{y}{z}$ met $x = t\cos t$, $y = t \sin t$, $z = t \tan t$

$$\frac{dF}{dt} = \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt}}_{(1)} + \underbrace{\frac{\partial f}{\partial y}\frac{dy}{dt}}_{(2)} + \underbrace{\frac{\partial f}{\partial z}\frac{dz}{dt}}_{(3)}$$

(1) $\left(\frac{y}{x} \cdot \frac{1}{y} + \frac{z}{x} \cdot \frac{1}{z}\right)\frac{dx}{dt} = \frac{2}{t\cos t}(\cos t - t \sin t)$

(2) $\left(\frac{y}{x} \cdot \frac{-x}{y^2} + \frac{z}{y} \cdot \frac{1}{z}\right)\frac{dy}{dt} = \left(\frac{-1}{t \sin t} + \frac{1}{t \sin t}\right)(t\cos t + \sin t)$

(3) $\left(\frac{z}{x} \cdot \frac{-x}{z^2} + \frac{z}{y} \cdot \frac{-y}{z^2}\right)\frac{dz}{dt} = \left(\frac{-1}{t \tan t} + \frac{-1}{t\tan t}\right)\left(\tan t + \frac{t}{\cos^2 t}\right)$

(1) + (2) + (3): $\cancel{\frac{2}{t}} - 2\tan t - \cancel{\frac{2}{t}} - \frac{2}{\tan t \cos^2 t}$

$= \frac{-2}{\sin t \cos t} - 2\tan t$

7 2) $f(x,y,z) = xy + yz + zx$ met $x = t^s$, $y = ts$, $z = t + s$

$$\frac{\partial f}{\partial t} = \underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}}_{1} + \underbrace{\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}_{2} + \underbrace{\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}}_{3}$$

1 $(y+z)st^{s-1} = (ts + t + s)st^{s-1} = t^s s^2 + t^s s + s^2 t^{s-1}$

2 $(x+z)s = (t^s + t + s)s = t^s s + ts + s^2$

3 $(y+x) = (t^s + ts)$

$= t^s(1+s)^2 + s^2 t^{s-1} + s^2 + 2ts$

$$\frac{\partial f}{\partial s} = \underset{1}{\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}} + \underset{2}{\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}} + \underset{3}{\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}}$$

1 $(ts + t + s)\, t^s \ln t = t^{s+1} s \ln t + t^{s+1}\ln t + t^s s \ln t$

2 $(t^s + t + s)\, t = t^{s+1} + t^2 + ts$

3 $(t^s + ts)$

$$\frac{\partial f}{\partial s} = t^{s+1}(s\ln t + \ln t + 1) + t^s(s\ln t + 1) + 2ts + t^2$$

3) $f(x,y) = x\ln y + y\ln x$ met $x = e^{t+s}$ en $y = e^{t-s}$

$$\frac{\partial f}{\partial t} = \underset{1}{\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}} + \underset{2}{\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}} + \underset{3}{\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}}$$

1 $\left(\ln y + \frac{y}{x}\right)e^{t+s} = \left(\ln e^{t-s} + \frac{e^{t-s}}{e^{t+s}}\right)e^{t+s} = (t - s + e^{-2s})e^{t+s}$

2 $\left(\frac{x}{y} + \ln x\right)e^{t-s} = \left(\frac{e^{t+s}}{e^{t-s}} + \ln e^{t+s}\right)e^{t-s} = (e^{2s} + t + s)e^{t-s}$

$\Rightarrow (t - s + 1)e^{t+s} + (t + s + 1)e^{t-s}$

$$\frac{\partial f}{\partial s} = \underset{1}{\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}} + \underset{2}{\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}} + \underset{3}{\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}}$$

1 $(t - s + e^{-2s})e^{t+s}$

2 $(e^{2s} + t + s)(-e^{t-s})$

$\Rightarrow (t - s - 1)e^{t+s} + (1 - t - s)e^{t-s}$

8 2) $\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{(x^2+y^2)^3}} \cdot 3(x^2+y^2)^2\, 2x = \frac{3x(x^2+y^2)^2}{(x^2+y^2)^{3/2}} = 3x\sqrt{x^2+y^2}$

$\frac{\partial^2 f}{\partial x^2} = 3\sqrt{x^2+y^2} + 3x \cdot \frac{1}{2}\frac{1}{\sqrt{}}\, 2x = \frac{3x^2 + 3y^2 + 3x^2}{\sqrt{}} = \frac{6x^2 + 3y^2}{\sqrt{x^2+y^2}}$

$\frac{\partial^2 f}{\partial x \partial y} = 3x \cdot \frac{1}{2}\frac{1}{\sqrt{}}\, 2y = \frac{3xy}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y} = 3y\sqrt{x^2+y^2} \Rightarrow \frac{6y^2 + 3x^2}{\sqrt{x^2+y^2}}$

4) $\frac{\partial f}{\partial x} = y + e^{1/y}$ $\quad \frac{\partial^2 f}{\partial x^2} = 0$ $\quad \frac{\partial^2 f}{\partial x \partial y} = 1 - \frac{e^{1/y}}{y^2}$

$\frac{\partial f}{\partial y} = x + xe^{1/y}\frac{-1}{y^2}$ $\quad \frac{\partial^2 f}{\partial y^2} = \frac{y^2(-xe^{1/y})\left(\frac{-1}{y^2}\right) + xe^{1/y} \cdot 2y}{y^4}$

$= \frac{xe^{1/y}(1+2y)}{y^4}$

3 i) $\ln\sqrt{x^2+y^2} - \arctan\frac{y}{x} = 0$

$$\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}} \cdot 2x\,dx + \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\, 2y\,dy$$

$$- \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}dy - \frac{y}{x^2}dx\right) = 0$$

$$\Leftrightarrow \left(\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\right)dx + \left(\frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\right)dy = 0$$

stel $dy = y'dx$

$$\Leftrightarrow \frac{x+y}{x^2+y^2}\,dx + \frac{y-x}{x^2+y^2}\,y'dx = 0$$

$$\Leftrightarrow y' = \frac{x+y}{x-y}$$

$$y'' = \frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2}$$

$$= \frac{(x-y)\left(\frac{x-y+x+y}{x-y}\right) - (x+y)\left(\frac{x-y-x-y}{x-y}\right)}{(x-y)^2}$$

$$= \frac{2x^2-2xy+2xy+2y^2}{(x-y)^3} = 2\frac{x^2+y^2}{(x-y)^3}$$

2) $y - x - \ln y = 0$

$$\Leftrightarrow dy - dx - \frac{1}{y}dy = 0$$

$$\Leftrightarrow (-1)dx + \left(1-\frac{1}{y}\right)dy = 0 \qquad \text{stel } dy = y'dx$$

$$\Leftrightarrow (-1)dx + \left(1-\frac{1}{y}\right)y'dx = 0$$

$$\Leftrightarrow y' = \frac{1}{1-\frac{1}{y}} = \frac{y}{y-1}$$

$$y'' = \frac{(y-1)y' - y\cdot y'}{(y-1)^2} = \frac{\frac{y^2-y-y^2}{y-1}}{(y-1)^2}$$

$$= -\frac{y}{(y-1)^3}$$

6 1) $2x\,dx + 2y\,dy - 2z\,dz - y\,dx - x\,dy = 0$

$\Leftrightarrow (2x - y)\,dx + (2y - x)\,dy - 2z\,dz = 0$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\Leftrightarrow \left(2x - y - 2z\frac{\partial z}{\partial x}\right)dx + \left(2y - x - 2z\frac{\partial z}{\partial y}\right)dy = 0$

$$\Leftrightarrow \begin{cases} 2x - y - 2z\dfrac{\partial z}{\partial x} = 0 \\ 2y - x - 2z\dfrac{\partial z}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{2x - y}{2z} \\ \dfrac{\partial z}{\partial y} = \dfrac{2y - x}{2z} \end{cases}$$

2) $xyz - x^2z - y^2z - z^2x - \ln(xyz) = 0$

$\Leftrightarrow \left(yz - 2xz - z^2 - \frac{1}{x}\right)dx + \left(xz - 2yz - \frac{1}{y}\right)dy$

$+ \left(xy - x^2 - y^2 - 2xz - \frac{1}{z}\right)dz = 0$

stel: $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$$\Leftrightarrow \begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{(xyz - 2x^2z - xz^2 - 1)z}{(xyz - x^2z - y^2z - 2xz^2 - 1)x} = \dfrac{z - xyz^2 + 2x^2z^2 + xz^3}{x^2yz - x^3z - xy^2z - 2x^2z^2 - x} \\ \dfrac{\partial z}{\partial y} = -\dfrac{(xyz - 2y^2z - 1)z}{(xyz - x^2z - y^2z - 2xz^2 - 1)y} = \dfrac{z + 2y^2z^2 - xyz^2}{xy^2z - x^2yz - y^3z - 2xyz^2 - y} \end{cases}$$

4) $(\cos y - z\sin x)dx + (-x\sin y + \cos z)dy + (-y\sin z + \cos x)dz = 0$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\Leftrightarrow \dfrac{\partial z}{\partial x} = \dfrac{-\cos y + z\sin x}{-y\sin z + \cos x}$ $\qquad \dfrac{\partial z}{\partial y} = \dfrac{x\sin y - \cos z}{\cos x - y\sin z}$ met $z = f(x,y)$

11 $\Phi(x,y,z) = \phi(ax + by + cz)$

2) $\dfrac{\partial \Phi}{\partial t} = \dfrac{\partial \phi}{\partial x}$

12 1) $\frac{\partial z}{\partial x} = 3x^2 - 12y \qquad \frac{\partial z}{\partial y} = 3y^2 - 12x$

$(3x^2 - 12y)dx + (3y^2 - 12x)dy = 0$

$\Leftrightarrow \frac{dx}{dy} = \frac{12y - 3x^2}{3y^2 - 12x} = \frac{4y - x^2}{y^2 - 4x}$

$\Rightarrow (6,6) \quad \frac{dx}{dy} = \frac{24 - 36}{36 - 24} = -1$

$y - y_1 = f'(x_1)(x - x_1)$

$\Leftrightarrow y - 6 = -1(x - 6)$

$\Leftrightarrow y = -x + 12$

2) $\nabla f(x,y) = \left(\underset{1}{\frac{\partial f}{\partial x}(x,y)}, \underset{2}{\frac{\partial f}{\partial y}(x,y)}\right)$

1) $\frac{\partial f}{\partial x} = 2(x^2+y^2)\cdot 2x - 32xy = 4\sqrt{3}(3+9) - 96\sqrt{3} = -48\sqrt{3}$

2) $\frac{\partial f}{\partial y} = 2(x^2+y^2)\cdot 2y - 16x^2 = 12(3+9) - 48 = 96$

$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \frac{3}{2\sqrt{3}}$

$\nabla f(x,y) = \left(4(x^3 + y^2x - 8xy), 4(x^2y + y^3 - 4x^2)\right)$

$\sim \left(1, \cancel{\frac{x^2y + y^3 - 4x^2}{x^3 + y^2x - 8xy}}\right) \qquad \neq \sim \left(1, \frac{\partial f/\partial y}{\partial f/\partial x}\right) \sim \left(1, -\frac{2}{\sqrt{3}}\right)$

$(\sqrt{3},3) \sim \left(1, \cancel{\frac{9 + 27 - 12}{\sqrt{3}^3 + 9\sqrt{3} - 24\sqrt{3}}}\right) = \left(1, \frac{24\sqrt{3}}{-18}\right) \cancel{= \left(1, -\frac{4\sqrt{3}}{3}\right)}$

$\nabla f(x,y) \perp$ raaklijn

$y - 3 = \frac{\sqrt{3}}{2}(x - \sqrt{3}) \Leftrightarrow y = \frac{\sqrt{3}}{2}x + \frac{3}{2}$

3) $(2(x^2+y^2)\,2x - 32x)dx + (2(x^2+y^2)\,2y + 32y)dy = 0$

$\Leftrightarrow \frac{dx}{dy} = -\frac{4(y(x^2+y^2) + 8y)}{4((x^2+y^2)x - 8x)} = -\frac{y(x^2+y^2) + 8y}{x(x^2+y^2) - 8x}$

$\Rightarrow \left(\frac{4\sqrt{2}}{3}, \frac{4}{3}\right) = -\frac{\frac{4}{3}\left(\frac{32}{9} + \frac{16}{9}\right) + \frac{32}{3}}{\frac{4\sqrt{2}}{3}\left(\frac{32}{9} + \frac{16}{9}\right) - \frac{32\sqrt{2}}{3}} = \frac{-\frac{128}{9}}{-\frac{32\sqrt{2}}{9}} = \frac{2\sqrt{2}}{2}$

4) $\nabla f\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \sim \left(1, \frac{\partial f/\partial y}{\partial f/\partial x}\right)$

$\frac{\partial f}{\partial x} = 2(x^2+y^2)2x - 4y \quad (1,1) = 4$

$\frac{\partial f}{\partial y} = 2(x^2+y^2)2y - 4x \quad (1,1) = 4$

$\nabla f(x,y) \sim (1,1) \to$ rico raaklijn $= -1$

$y - 1 = -1(x-1) \Leftrightarrow y = -x+2$

5) $(6x^5 - 24x^3 + 24x)dx + 8y\,dy = 0$

$\Leftrightarrow \frac{dy}{dx} = \frac{-8y}{6x^5 - 24x^3 + 24x}$

$(1, \frac{1}{2}) = \frac{-4}{6} = -\frac{3}{2}$

$y - \frac{1}{2} = -\frac{2}{3}(x-1)$

13 1) $z = f(x,y) = y^4 - 4x^4 - 6xy$

$P(1,2) = 16 - 4 - 12 = 0$ niveaulijn

$(-16x^3 - 6y)dx + (4y^3 - 6x)dy = 0$

$\Leftrightarrow \frac{dx}{dy} = \left(\frac{4y^3-6x}{16x^3+6y}\right)^{-1} = \left(\frac{2y^3-3x}{8x^3+3y}\right)^{-1}$

$(1,2) \Rightarrow \frac{dx}{dy} = \left(\frac{16-3}{8+6}\right)^{-1} = \left(\frac{13}{14}\right)^{-1} = \frac{14}{13}$

$y - 2 = \frac{14}{13}(x-1) \Leftrightarrow 13y - 14x - 12$

2) $P(1,0)$: $\ln 1 + e^0 = 1$

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \sim \left(1, \frac{\partial f/\partial y}{\partial f/\partial x}\right)$

$= \left(1, \frac{-\frac{1}{x}e^{-y/x}}{\frac{1}{x} - \frac{y}{x^2}e^{-y/x}}\right) \quad (1,0) \Rightarrow \left(1, \frac{1}{1}\right)$

$y = x - 1$

14 homogeen: $f(\lambda x, \lambda y) = \lambda^k f(x,y)$

Euler: $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = k\, f(x,y)$

1) $f(\lambda x, \lambda y) = (\lambda x \lambda y)^{1/2}$

$= \lambda f(x,y) \rightarrow$ homogeen graad 1

Euler: $x \cdot \frac{1}{2} \frac{1}{\sqrt{xy}} + y \frac{1}{2} \frac{1}{\sqrt{xy}} = f(x,y)$

$\Leftrightarrow \frac{xy}{\sqrt{xy}} = \sqrt{xy}$

$\Leftrightarrow \sqrt{xy} = \sqrt{xy}$ ok

2 $f(tx,ty) = (tx)^3 - (tx)(ty) + (ty^3)$

$= t^3x^3 - t^2xy + t^3y \rightarrow$ niet homogeen

4) $f(tx, ty, tz) = \frac{(tx)(ty)^2}{(tz)} + 2(tx)(tz)$

$= \frac{t^3xy^2}{tz} + 2t^2xz$

$= t^2 \frac{xy^2}{z} + t^2\, 2xz \rightarrow$ homogeen graad 2

Euler: $x\left(\frac{y^2}{z} + 2z\right) + y\left(\frac{2xy}{z}\right) + z\left(-\frac{xy^2}{z^2} + 2x\right) = 2\left(\frac{xy}{z}^2 + 2xz\right)$

$\Leftrightarrow \cancel{\frac{xy^2}{z}} + 2xz + \frac{2xy^2}{z} - \cancel{\frac{xy^2}{z}} + 2xz$

$\Leftrightarrow 2\left(\frac{xy^2}{z} + 8xz\right)$

15 $Q = a + bP^2 + R^{1/2}$ $a<0,\ b>0$ P: prijs R: hoeveelheid

1) $\frac{EQ}{EP} = \frac{P}{Q} \frac{\partial Q}{\partial P} = \frac{P}{Q} 2bP = \frac{2bP^2}{Q}$

$\frac{EQ}{ER} = \frac{R}{Q} \cdot \frac{1}{2}\frac{1}{\sqrt{R}} = \frac{R^{1/2}}{2Q}$

2) $d(\varepsilon_{QP}) = d\left(\frac{2bP^2}{Q}\right) = 2b\, d\left(\frac{P^2}{Q}\right)$

$= 2b\left[\frac{\partial}{\partial P}\left(\frac{P^2}{Q}\right) dP + \frac{\partial}{\partial Q}\left(\frac{P^2}{Q}\right) dQ\right]$

$= 2b\left[\frac{2P}{Q} dP - \frac{P^2}{Q^2} dQ\right]$ $\rightarrow dQ = \frac{\partial Q}{\partial R} dR + \frac{\partial Q}{\partial P} dP = \frac{1}{2} R^{1/2} dR + 2bP\,dP$

$$= 2b\left[\frac{2P}{Q}dP - \frac{P^2}{Q^2}\left(\frac{1}{2}R^{-1/2}dR + 2bP\,dP\right)\right]$$

$$= 2b\left[\left(\frac{2P}{Q} - 2b\frac{P^3}{Q^2}\right)dP - \frac{1}{2}\frac{P^2}{Q^2R^{1/2}}dR\right]$$

$$= 2b\left[\frac{2P}{Q}\underbrace{\left(1 - \frac{bP^2}{Q}\right)}_{\frac{Q-bP^2}{Q} = \frac{a+R^{1/2}}{Q}}dP - \frac{1}{2}\frac{P^2}{Q^2R^{1/2}}dR\right]$$

$$= 2b\left[\frac{2P}{Q}(a + R^{1/2})dP - \frac{1}{2}\frac{P^2}{Q^2R^{1/2}}dR\right]$$

$$d\varepsilon_{QR} = d\left(\frac{R^{1/2}}{2Q}\right) = \frac{1}{2}d\left(\frac{R^{1/2}}{Q}\right)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial R}\left(\frac{R^{1/2}}{Q}\right)dR + \frac{\partial}{\partial Q}\left(\frac{R^{1/2}}{Q}\right)dQ\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{1}{QR^{1/2}}dR - \frac{R^{1/2}}{Q^2}dQ\right]$$

$$= \frac{1}{2}\left[\frac{1}{2QR^{1/2}}dR - \frac{R^{1/2}}{Q^2}\left(\frac{1}{2}R^{-1/2}dR + 2bP\,dP\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2QR^{1/2}} - \frac{1}{2Q^2}\right)dR - \frac{2bPR^{1/2}}{Q^2}dP\right]$$

$$= \frac{1}{2}\left(\frac{Q - R^{1/2}}{2Q^2R^{1/2}}dR - \frac{2bPR^{1/2}}{Q^2}dP\right)$$

$$= -\frac{bPR^{1/2}}{Q^2}dP + \frac{1}{4Q^2R^{1/2}}(a + bP^2)dR$$

16 X: vraag Y_f: inkomen P: prijs $X = Y_f^{1/2} + P^{-2}$

$$\varepsilon_{XP} = \frac{\varepsilon X}{\varepsilon P} = \frac{P}{X}\frac{\partial X}{\partial P} = \frac{P}{X}\cdot(-2)P^{-3} = \frac{-2}{P^2(X)}$$

Werkcollege H2

1 $f(x,y) = \ln(x^2+y^2) - 3xy \qquad P(1,0)$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - 3y \Rightarrow (1,0): \ 2$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} - 3x \Rightarrow (1,0): \ -3$$

$$t_p: \ y = +\frac{2}{3}(x-1)$$

$$n_p: \ y = \frac{-3}{2}(x-1)$$

2 $z(x,y) = \frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{4} = 1$

a) $x^2 + 4y^2 + 4z^2 = 4$

$$\cancel{\frac{\partial z}{\partial x} = 2x + 4 \cdot 2 \frac{\partial z}{\partial x} = 2x + 16x = 18x}$$

$$\cancel{\frac{\partial z}{\partial y} = 8y + 4 \cdot 2 \frac{\partial z}{\partial y} = 8y + 64y = 72y}$$

$$\cancel{\frac{\partial f}{\partial x} = 2x} \qquad \cancel{\frac{\partial f}{\partial y} = 8y} \qquad \cancel{\frac{\partial f}{\partial z} = 8z}$$

$$x = 2 \Rightarrow \frac{y^2}{4} + \frac{z^2}{4} - \frac{3}{4} = 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}y = \frac{-\sqrt{2}}{2} \qquad \frac{\partial f}{\partial z} = \frac{1}{2}z = \frac{1}{2}$$

$$-\frac{\partial f/\partial z}{\partial f/\partial y} = \frac{+\sqrt{2}/2}{\cancel{\sqrt{2}}/2} = \frac{\sqrt{2}}{\cancel{2}}$$

$$z - 1 = \sqrt{2}\,(y + \sqrt{2})$$

b) $y = -\sqrt{2} \Rightarrow \frac{x^2}{16} + \frac{1}{2} + \frac{z^2}{4} = 1 \Leftrightarrow \frac{x^2}{16} + \frac{z^2}{4} - \frac{1}{2} = 0$

$$\frac{\partial f}{\partial x} = \frac{x}{8} = \frac{1}{4} \qquad \frac{\partial f}{\partial z} = \frac{1}{2}z = \frac{1}{2}$$

$$\text{rico}: \ \frac{-1/4}{1/2} = -\frac{1}{2}$$

$$z - 1 = \frac{-1}{2}(x-2)$$

~~c)~~ $\cancel{z = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} - \frac{3}{4} = 0}$

$$\cancel{\frac{\partial f}{\partial x} = \frac{x}{8} = \frac{1}{4} \qquad \frac{\partial f}{\partial y} = \frac{-\sqrt{2}}{2}}$$

$$\cancel{\text{rico}: \ \frac{1/4}{\sqrt{2}/2} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}}$$

c) $\frac{\partial z}{\partial v}(p) = v \cdot \nabla f \left(\frac{\partial z}{\partial x} , \frac{\partial z}{\partial y} \right)$

$$x^2 + 4y^2 + 4z^2 = 4$$

$$2x\,dx + 8y\,dy + 8z\,dz = 0$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Leftrightarrow \left(2x + 8z \frac{\partial z}{\partial x}\right) dx + \left(8y + 8z \frac{\partial z}{\partial y}\right) dy = 0$$

$$\begin{cases} 2x + 8z \frac{\partial z}{\partial x} = 0 \\ 8y + 8z \frac{\partial z}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{\partial z}{\partial x} = \frac{-x}{4z} \\ \frac{\partial z}{\partial y} = \frac{-y}{z} \end{cases}$$

$$\Rightarrow \frac{\partial z}{\partial v}(p) = v \cdot \nabla f \left(\frac{-x}{4z} , \frac{-y}{z} \right)$$

$$= (2, -1) \cdot \left(\frac{-2}{4} , \sqrt{2} \right)$$

$$= -1 - \sqrt{2}$$

d) $\begin{cases} x = 2 + 2t \\ y = -1 - \sqrt{2}\,t \\ z = 1 + (-1 - \sqrt{2})t \end{cases}$

3 $w(x,y) = \ln(x^2 + y^2)$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du}$$

$$= \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2} \cdot 3$$

$$\Rightarrow \frac{2(u+2v+1)}{(u+2v+1)^2 + (3u-v-1)^2} + \frac{6(3u-v-1)}{\dots}$$

$$\Rightarrow \left(\frac{4}{7}, \frac{-2}{7}\right) \quad \frac{2}{1+1} + \frac{6}{2} = 1 + 3 = 4$$

4 $V = C(Y) + I + G + X(I) - M(I, Y)$

c) $\cancel{\frac{\partial V}{\partial G}} \Rightarrow V = Y^3 + 5Y + I + G + \frac{I}{2} - (3Y - I)^2$

$$\frac{\partial V}{\partial G} = 3Y^2 dY + 5dy + 1 - 6(3Y - I)\,dY \qquad dY = \frac{1}{2\sqrt{1}} + \frac{1}{2\sqrt{4}} = \frac{3}{4}$$

klopt niet

$$= 3 \cdot 9 \cdot \frac{3}{4} + 5 \cdot \frac{3}{4} + 1 - 6(3 \cdot 3 - 1) \cdot \frac{3}{4}$$

$$= 81/4 + 15/4 + 4/4 - 144/4 =$$

5 $e^{x+y-y^2} = e^{x+y-y^2}(1-2y)$

$\Leftrightarrow (1-2y) = 1$

$\Leftrightarrow y$

$ye^{\cdots} = xe^{\cdots}(1-2y)$

$\Leftrightarrow y = x(1-2y)$

$\Leftrightarrow x = \dfrac{y}{1-2y}$

$\dfrac{y^2}{(1-2y)^2} + y^2 = 2$

$\Leftrightarrow y^2 + y^2(1\cancel{*} - 4y + y^2) = 2$

$\Leftrightarrow y^2(1 + 1 - 4y + y^2) = 2$

$\Leftrightarrow y^2(y^2 - 4y + 2) - 2 = 0$

$\Leftrightarrow y^4 - 4y^3 + 2y^2 - 2 = 0$

6 $1 - x^2 - y^2 > 0$ $\qquad x - y^2 \geq 0$

$\Leftrightarrow x^2 + y^2 < 1 \quad \wedge \quad x - y^2 \geq 0$

$$\left(\frac{-2x}{1-x^2-y^2} \cancel{dx} - \frac{1}{2\sqrt{x-y^2}}\right)dx + \left(\frac{-2y}{1-x^2-y^2} + \frac{2y}{2\sqrt{x-y^2}}\right)dy$$

Optimalisatie van functies van meerdere veranderlijken

1 A $A_1 = 2 > 0$ $A_2 = 2 + 1 = 3 > 0$ → pos. definiet

B $B_1 = -3 < 0$ $B_2 = 15 - 16 = -1 < 0$ → indefiniet

C $C_1 = 1 > 0$ $C_2 = 4 - 4 = 0$ $C_3 = (24-25) - 2 \cdot 12 = -1 - 24 = -25 < 0$
→ indefiniet

D $D_1 = 1 > 0$ $D_2 = 2 > 0$ $D_3 = (4-9) = \cancel{-5} - 10 < 0$ → indefinie t

2 1 $x_1^2 + 2x_1x_2 - x_2^2$
1ste rij, 2de kolom half coefficiënt

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$A_1 = 1 > 0$ $A_2 = -2 < 0$ → indefiniet

2 $\underset{\sim}{A} = \begin{pmatrix} -4 & -1 \\ -1 & -1 \end{pmatrix}$ $A_1 = -4 < 0$
$A_2 = 4 - 1 = 3 > 0$
→ neg. definiet

3 $\underset{\sim}{A} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ $A_1 = 1 > 0$
$A_2 = 0$
$A_3 = -4 + 2 = -2 < 0$ → indefiniet

3 1) $\frac{\partial f}{\partial x} = y - 2(x^2+y) \cdot 2x = 0$ $\begin{cases} y - 4x(x^2+y) = 0 & (1) \\ x - 2(x^2+y) = 0 \quad (\cdot 2x) & (2) \end{cases}$

$\frac{\partial f}{\partial y} = x - 2(x^2+y) = 0$

$\Leftrightarrow y - 4x(x^2+y^2) = 2x^2 - 4x(x^2+y^2) = 0$

$\Leftrightarrow y = 2x^2$ (3)

(3) in (1) $2x^2 - 4x(x^2 + 2x^2) = 0$

$\Leftrightarrow x^2 - 2x^3 - 4x^3 = 0$

$\Leftrightarrow -6x^3 + x^2 = 0$

$\Leftrightarrow x^2(-6x + 1) = 0$

$\Leftrightarrow x = 0 \cup x = 1/6$

$y = 0 \cup y = 1/18$

$(0,0) \cup (\frac{1}{6}, \frac{1}{18})$

$$H(x,y) = \begin{vmatrix} -4(x^2+y)-8x^2 & 1-4x \\ 1-4x & -2 \end{vmatrix}$$

$(0,0)$ $H_1(0,0) = 0$ $H_2(0,0) = -2-1 = -3 < 0$

$\rightarrow$ zadelpunt

$\left(\frac{1}{6}, \frac{1}{18}\right)$ $H_1 = \frac{-1}{3} - \frac{2}{9} = -\frac{5}{9} < 0$

$H_2 = \frac{10}{9} - \frac{1}{9} = 1 > 0$

$\rightarrow$ max.

2 $\frac{\partial f}{\partial x} = 2x = 0$ $\begin{cases} x = 0 \\ y = 1 \end{cases}$ $(0,1)$

$\frac{\partial f}{\partial y} = 2(y-1) = 0$

$H(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$ $H_1 = 2 > 0$ $H_2 = 4 > 0$ $\rightarrow$ min

3 $\begin{cases} \frac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0 & (1) \\ \frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 & (2) \end{cases}$

$\Leftrightarrow \cos x = \cos y$

$\Leftrightarrow x = y$ (3)

(3) in (2)

$\cos y + \cos 2y = 0$

$\Leftrightarrow \cos y = -\cos 2y$

4 $\begin{cases} \frac{\partial f}{\partial x} = 2x - 2 = 0 \\ \frac{\partial f}{\partial y} = -2y + 4 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$

$$H(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}$$

$H_1 = 2 > 0$
$H_2 = -4 < 0$ $\rightarrow$ zadelpunt

4 $\begin{cases} \frac{\partial f}{\partial x} = 2x - 6y + 3 = 0 \\ \frac{\partial f}{\partial y} = -6x + 18y + a = 0 \end{cases}$

$\Leftrightarrow \begin{cases} 2x - 6y = 3 \\ -6x + 18y = -a \end{cases}$

$\rightarrow$ ∞ veel oplossingen

rang coefficiëntenmatrix = 1 $\Rightarrow$ rang aangevulde matrix = 1

$$\begin{array}{cc} A & B \\ \left(\boxed{\begin{matrix} 2 & -6 \\ -6 & 18 \end{matrix}} \right| & \left. \boxed{\begin{matrix} -3 \\ -a \end{matrix}} \right) \end{array} \Rightarrow a = -9$$

$\det \underset{\sim}{A} = 0$

5 $f(x,y) = (ax^2 - by^2)e^{-(x^2+y^2)}$ $\quad a \neq 0$ en $b \neq 0$

$$\begin{cases} \frac{\partial f}{\partial x} = (ax^2 - by^2)e^{-(x^2+y^2)} \cdot (-2x) + 2ax \cdot e^{-(x^2+y^2)} = 0 & (1) \\ \frac{\partial f}{\partial y} = (ax^2 - by^2)e^{-(x^2+y^2)} \cdot (-2y) + (-2by)e^{-(x^2+y^2)} = 0 & (2) \end{cases}$$

$\begin{cases} 2ax - 2x(ax^2 - by^2) = 0 & \cdot y \\ -2by - 2y(ax^2 - by^2) = 0 & \cdot x \end{cases}$ $\quad -2y$

$2axy - 2xy(\quad) = -2bxy - 2xy(\quad)$

$\Leftrightarrow axy = -bxy$ $\quad$ $\rightarrow$ OK $a = -b$

$\Leftrightarrow a = -b$

$$\left(\begin{cases} 2x(a - ax^2 + by^2) \cdot e^{-(x^2+y^2)} = 0 \\ -2y(b + ax^2 - by^2) \cdot e^{-(x^2+y^2)} = 0 \end{cases} \right)$$

6 $16q = 65 - 2(q_1-5)^2 - 4(q_2-4)^2$ $p_1 = 8$ en $p_2 = 4$ $p = 32$

1 $\omega(q) = p \cdot q - p_1 q_1 - p_2 q_2$

$= 32q - 8q_1 - 4q_2$ $32q = 2 \cdot 16q$

$\Rightarrow \omega(q_1, q_2) = 130 - 4(q_1-5)^2 - 8(q_2-4)^2 - 8q_1 - 4q_2$

2 $\begin{cases} \frac{\partial \omega}{\partial q_1} = -8(q_1-5) - 8 = 0 & \Leftrightarrow q_1 = 4 \\ \frac{\partial \omega}{\partial q_2} = -16(q_2-4) - 4 = 0 & \Leftrightarrow q_2 = \frac{15}{4} \end{cases}$

$H = \begin{vmatrix} -8 & 0 \\ 0 & -16 \end{vmatrix}$ $H_1 = -8 < 0$

$H_2 = 128 > 0$

$\to$ max.

3 $\omega(4, \frac{15}{4}) = \frac{157}{2} = 78{,}5$

7 $\omega(q_1, q_2) = p_1 q_1 + p_2 q_2 - K(q_1, q_2)$

$= (20-q_1)q_1 + (40-5q_2)q_2 - q_1^2 - 2q_1q_2 - 3q_2^2$

$= 20q_1 - q_1^2 + 40q_2 - 5q_2^2 - q_1^2 - 2q_1q_2 - 3q_2^2$

$= -2q_1^2 + 20q_1 - 2q_1q_2 - 8q_2^2 + 40q_2$

$\begin{cases} \frac{\partial \omega}{\partial q_1} = -4q_1 + 20 - 2q_2 = 0 & \Leftrightarrow q_1 = 5 - \frac{1}{2}q_2 \quad (1) \\ \frac{\partial \omega}{\partial q_2} = -2q_1 - 16q_2 + 40 = 0 & (2) \end{cases}$

(1) in (2) $-10 + q_2 - 16q_2 + 40 = 0$

$\Leftrightarrow -15q_2 = -30$

$\Leftrightarrow \begin{cases} q_2 = 2 \\ q_1 = 4 \end{cases}$ $H = \begin{vmatrix} -4 & -2 \\ -2 & -2 \end{vmatrix}$ $H_1 = -4 < 0$

$H_2 = 4 > 0$

$\to$ max

$p_1 = 20 - q_1$
$= 16$

$p_2 = 40 - 5q_2$
$= 30$

8 1 $\mathcal{L}(x,y,\lambda) = 5x^2 + 6y^2 - xy + \lambda(x+2y-24)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 10x - y + \lambda = 0 \quad \cdot 2 \quad \Leftrightarrow \\ \frac{\partial \mathcal{L}}{\partial y} = 12y - x + 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x + 2y - 24 = 0 \end{cases}$$

$\Leftrightarrow 20x - 2y + 2\lambda = 12y - x + 2\lambda$

$\Leftrightarrow 21x - 14y = 0$

$\Leftrightarrow 3x - 2y = 0$

$\Leftrightarrow y = \frac{3}{2}x$

$x + 2 \cdot \frac{3}{2}x - 24 = 0$

$\Leftrightarrow 4x - 24 = 0$

$\Leftrightarrow \begin{cases} x = 6 \\ y = 9 \end{cases}$

$$H^* = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 10 & -1 \\ 2 & -1 & 12 \end{vmatrix} = \cancel{(-1)(120+1) + 2(-1-20)}$$
$$= \cancel{-121} \; (-1)(12+2) + 2(-1-20)$$
$$= -14 - 42 = -56 < 0 \rightarrow \text{min}$$

Bes: $(6,9)$ is min.

2 $\mathcal{L}(x,y,\lambda) = 12xy - 3y^2 - x^2 + \lambda(x+y-16)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 12y - 2x + \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = 12x - 6y + \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 16 = 0 \end{cases}$$

$12y - 2x = 12x - 6y$

$\Leftrightarrow 6y - x = 6x - 3y$

$\Leftrightarrow 7x = +9y$

$\Leftrightarrow x = +\frac{9}{7}y$

$+\frac{9}{7}y + y - 16 = 0$

$\Leftrightarrow \frac{16}{7}y = 16$

$\Leftrightarrow \begin{cases} y = 7 \\ x = 9 \end{cases}$

$$H^* = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 12 \\ 1 & 12 & -6 \end{vmatrix} = (-1)(-6-12) + (12+2)$$
$$= 18 + 14 = 32 > 0 \rightarrow \text{max}$$

Bes: $(9,7)$ is max.

3 $\mathcal{L}(x,y,z) = 3x^2 - 4y^2 - xy + \lambda(2x+y-21)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 6x - y + 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = -8y - x + \lambda = 0 \quad \cdot 2 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 2x + y - 21 = 0 \end{cases}$$

$6x - y + 2\lambda = -16y - 2x + 2\lambda$

$\Leftrightarrow 8x = -15y$

$\Leftrightarrow y = -\frac{8}{15}x$

$2x + \frac{-8}{15}x - 21 = 0$

$\Leftrightarrow \frac{\quad}{15}x = 21$

$\Leftrightarrow x =$

9 Opl 1: f stijgt snelst in richting van $\nabla f(1,1)$

$\nabla f\left(6x^2 + 5y^2 - \frac{1}{xy}y,\ 10xy - \frac{x}{xy}\right)$

$\nabla f(1,1) = (10, 9)$

$\rightarrow$ bijhorende eenheidsvector $\left(\frac{10}{\sqrt{181}}, \frac{9}{\sqrt{181}}\right)$

Opl 2: via extremum probleem

$(\alpha, \beta) = \underset{\sim}{v}$ eenheidsvectoren: $\alpha^2 + \beta^2 = 1$

$\frac{\partial f}{\partial \underset{\sim}{v}}(1,1) = \underset{\sim}{v} \cdot \nabla f(1,1) = (\alpha, \beta) \cdot (10, 9)$

$= 10\alpha + 9\beta$

nevenvoorwaarde $\alpha^2 + \beta^2 - 1 = 0$

$\mathcal{L}(\alpha, \beta, \lambda) = 10\alpha + 9\beta + \lambda(\alpha^2 + \beta^2 - 1)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \alpha} = 10 + 2\alpha\lambda = 0 \quad \cdot \beta \\ \frac{\partial \mathcal{L}}{\partial \beta} = 9 + 2\beta\lambda = 0 \quad \cdot \alpha \\ \frac{\partial \mathcal{L}}{\partial \lambda} = \alpha^2 + \beta^2 - 1 = 0 \end{cases}$$

$10\beta + = 9\alpha$

$\Leftrightarrow \alpha = \frac{10}{9}\beta$

$\frac{100}{81}\beta^2 + \beta^2 = 1$

$\Leftrightarrow \beta = \sqrt{\frac{81}{181}} = \frac{\pm 9}{\sqrt{181}}$

$\alpha = \sqrt{\frac{100}{181}} = \frac{\pm 10}{\sqrt{181}}$

$\lambda = \frac{-5}{\alpha}$

$= \pm \frac{\sqrt{181}}{2}$

$$\underset{\sim}{H} = \begin{pmatrix} 0 & 2\alpha & 2\beta \\ 2\alpha & 2\lambda & 0 \\ 2\beta & 0 & 2\lambda \end{pmatrix} \leftarrow \qquad \det \underset{\sim}{H} = -2\alpha(4\alpha\lambda) + 2\beta(-4\alpha\lambda)$$

$$= (-8\alpha^2 - 8\beta^2)\lambda$$

$$= -8\lambda$$

① kritisch punt $\left(\frac{10}{\sqrt{181}}, \frac{9}{\sqrt{181}}, -\frac{\sqrt{181}}{2}\right) \Rightarrow \det \underset{\sim}{H} > 0$

$\rightarrow$ MAX snelst stijgt

② $\left(\frac{-10}{\sqrt{181}}, \frac{-9}{\sqrt{181}}, \frac{\sqrt{181}}{2}\right) \Rightarrow \det \underset{\sim}{H} < 0$ $\rightarrow$ MIN snelst daalt

10 $U(q_1, q_2) = q_1 q_2^2 - 10 q_1$ $\qquad 116 = p_1 q_1 + p_2 q_2$

$= 2q_1 + 8q_2$

$$\mathcal{L}(q_1, q_2, \lambda) = q_1 q_2^2 - 10 q_1 + \lambda(2q_1 + 8q_2 - 116)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial q_1} = q_2^2 - 10 + 2\lambda = 0 & \cdot 4 \\ \frac{\partial \mathcal{L}}{\partial q_2} = 2q_1 q_2 + 8\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 2q_1 + 8q_2 - 116 = 0 & (1) \end{cases}$$

$$4q_2^2 - 40 = 2q_1 q_2$$

$$\Leftrightarrow q_1 = \frac{2q_2^2 - 20}{q_2} \quad (2)$$

(2) in (1) $\quad \frac{4q_2^2 - 40}{q_2} + 8\frac{q_2^2}{q_2} = 116$

$$\Leftrightarrow 12q_2^2 - 116q_2 - 40 = 0$$

$$\Leftrightarrow 6q_2^2 - 58q_2 - 20 = 0 \qquad \sqrt{D} = 62$$

$$q_2 = \frac{58 + 62}{12} = 10 \Rightarrow q_1 = 18$$

$q_2 = -1/3$ $\rightarrow$ eco. niet relevant

$q_2 = 0$

$\begin{cases} \lambda = 5 \\ \lambda = 0 \end{cases}$

stijgdig

$$H^* = \begin{vmatrix} 0 & 2 & 8 \\ 2 & 0 & 2q_2 \\ 8 & 2q_2 & 2q_1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 8 \\ 2 & 0 & 20 \\ 8 & 20 & 36 \end{vmatrix} = (-2)(-88) + 8 \cdot 40$$

$$= 496 > 0$$

Bes: $(q_1, q_2) = (18, 10)$ is max

$U(18, 10) = 1620$

11 $U(q_1,q_2)=q_1^2q_2$

1 $U(2,2)=8$

vgl: $q_1^2q_2=8$ met nutsniveau 8

2 $d(q_1^2q_2)=d(8)$

$\Leftrightarrow 2q_1q_2\,dq_1+q_1^2\,dq_2=0$

$\Leftrightarrow \dfrac{dq_2}{dq_1}=-\dfrac{2q_1q_2}{q_1^2}=-\dfrac{2q_2}{q_1}\overset{(2,2)}{=}-2$

$q_2-2=-2(q_1-2)\Leftrightarrow q_2=-2q_1+6$

3 $180=6q_1+5q_2$

$\mathcal{L}(q_1,q_2,\lambda)=q_1^2q_2+\lambda(6q_1+5q_2-180)$

$$\begin{cases}\dfrac{\partial\mathcal{L}}{\partial q_1}=2q_1q_2+6\lambda=0\\ \dfrac{\partial\mathcal{L}}{\partial q_2}=q_1^2+5\lambda=0 \quad \cdot\dfrac{6}{5}\\ \dfrac{\partial\mathcal{L}}{\partial\lambda}=6q_1+5q_2-180=0 \quad (1)\end{cases}$$

$2q_1q_2=\frac{6}{5}q_1^2$

$\Leftrightarrow q_2=\dfrac{3q_1^2}{5q_1}=\dfrac{3}{5}q_1 \quad (2)$

(2) in (1) $6q_1+3q_1=180$

$\Leftrightarrow \begin{cases}q_1=20\\ q_2=12\end{cases}$

$$H^*=\begin{vmatrix}0&6&5\\6&2q_2&2q_1\\5&2q_1&0\end{vmatrix}=\begin{vmatrix}0&6&5\\6&24&40\\5&40&0\end{vmatrix}=(-6)(-200)+5(240-120)=1200+600=1800>0$$

Bes: $(20,12)$ is max

$U(20,12)=4800$

4 $\mathcal{L}(q_1,q_2,\lambda)=q_1^2q_2+\lambda(6q_1+5q_2-b)$

$q_2=\frac{3}{5}q_1$

$\Rightarrow 6q_1+3q_1-b=0$

$\Leftrightarrow q_1=\dfrac{b}{9} \qquad q_2=\dfrac{b}{15} \qquad b>0$

12 $U(q_1, q_2) = q_1^2 + 3q_2^2 + q_1 q_2$

1 vgl: $q_1^2 + 3q_2^2 + q_1 q_2 = s$

2 $d(\ldots) = d(s)$

$\Leftrightarrow (2q_1 + q_2)dq_1 + (6q_2 + q_1)dq_2 = 0$

$\Leftrightarrow \dfrac{dq_2}{dq_1} = -\dfrac{2q_1 + q_2}{6q_2 + q_1} \overset{(1,1)}{=} -\dfrac{3}{7}$

$q_2 - 1 = -\dfrac{3}{7}(q_1 - 1)$

$\Leftrightarrow 7q_2 = -3q_1 + 10$

$\Rightarrow 7q_2 + 3q_1 = 10$

3 $150 = 3q_1 + 6q_2$

$\mathcal{L} = q_1^2 + 3q_2^2 + q_1 q_2 + \lambda(3q_1 + 6q_2 - 150) =$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial q_1} = 2q_1 + q_2 + 3\lambda = 0 & \cdot 2 \\ \dfrac{\partial \mathcal{L}}{\partial q_2} = 6q_2 + q_1 + 6\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = 3q_1 + 6q_2 - 150 = 0 & (1) \end{cases}$$

$4q_1 + 2q_2 = 6q_2 + q_1$

$\Leftrightarrow 3q_1 = 4q_2$

$\Leftrightarrow q_1 = \frac{4}{3} q_2 \quad (2)$

(2) in (1) $4q_2 + 6q_2 - 150 = 0$

$\Leftrightarrow \begin{cases} q_2 = 15 \\ q_1 = 20 \end{cases}$

$$H^* = \begin{vmatrix} 0 & 3 & 6 \\ 3 & 2 & 1 \\ 6 & 1 & 6 \end{vmatrix} = (-3)(18-6) + 6(3-12)$$

$= -36 - 54$

$= -90 < 0$

$\rightarrow$ Min

4 nevenvoorwaarde: $4q_1 + p_2 q_2 - b = 0$ optimum: $(2,4)$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial q_1} = 2q_1 + q_2 + 4\lambda = 0 & (2,4) & \Rightarrow & 8 + 4\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial q_2} = 6q_2 + q_1 + p_2\lambda = 0 & (\;) & \Rightarrow & 26 + p_2\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = 4q_1 + p_2 q_2 - b = 0 & (\;) & \Rightarrow & 8 + 4p_2 - b = 0 \end{cases}$$

$$\begin{cases} \lambda = -2 \\ p_2 = 13 \\ b = 60 \end{cases}$$

14 $D = 135$ $\quad q = z_2 + z_3 \quad \frac{T_2}{z_2} = \frac{T_3}{z_3} = \frac{T_2 + T_3}{z_2 + z_3}$

$C_2 = 120 \quad \frac{D}{q} = \frac{D}{z_2 + z_3}$

$C_3 = 150$

$C_1 = 45$

1 TK = bestelkosten + beheerskosten + voorraadtekortkosten
 1 2 3

1) $C_1 \cdot \frac{D}{q} = \frac{C_1 \cdot D}{z_2 + z_3} = \frac{6075}{z_2 + z_3}$

2) $C_2 \cdot \frac{T}{T_2 + T_3} \cdot \frac{1}{2}\left(\frac{T_2}{T} \cdot z_2\right)$

$= \frac{1}{2} C_2 z_2 \cdot \frac{z_2}{z_2 + z_3} = \frac{1}{2} C_2 \frac{z_2^2}{z_2 + z_3} = \frac{60 z_2^2}{z_2 + z_3}$

3) $\frac{1}{2} C_3 \frac{z_3^2}{z_2 + z_3} = \frac{75 z_3^2}{z_2 + z_3}$

$K(z_2, z_3) = \frac{6075}{z_2 + z_3} + \frac{60 z_2^2}{z_2 + z_3} + \frac{75 z_3^2}{z_2 + z_3}$

2 $\begin{cases} \frac{\partial K}{\partial z_2} = \frac{-6075}{(z_2+z_3)^2} + \frac{(z_2+z_3)120 z_2 - 60 z_2^2}{(z_2+z_3)^2} + \frac{-75 z_3^2}{(z_2+z_3)^2} = 0 \\ \quad = (z_2+z_3)120 z_2 - (6075 + 60 z_2^2 + 75 z_3^2) = 0 \quad (1) \\ \frac{\partial K}{\partial z_3} = \frac{-6075}{(z_2+z_3)^2} - \frac{60 z_2^2}{(z_2+z_3)^2} + \frac{(z_2+z_3)150 z_3 - 75 z_3^2}{(z_2+z_3)^2} = 0 \\ \quad = (z_2+z_3)150 z_3 - (6075 + 60 z_2^2 + 75 z_3^2) = 0 \quad (2) \end{cases}$

(1) − (2) $\begin{cases} 120 z_2 - 150 z_3 = 0 \\ (120 z_2 + 150 z_3)(z_2 + z_3) - 2(6075 + 60 z_2^2 + 75 z_3^2) = 0 \end{cases}$
(1) + (2)

$\Leftrightarrow z_2 = \frac{5}{4} z_3$

$(150 z_3 + 150 z_3)(\frac{5}{4} z_3 + z_3) - 2(6075 + \frac{375}{4} z_3^2 + 75 z_3^2) = 0$

$\Leftrightarrow 675 z_3^2 - 12150 - \frac{675}{2} z_3^2 = 0$

$\Leftrightarrow z_3^2 = 36$

$\Leftrightarrow \begin{cases} z_3 = 6 \\ z_2 = \frac{15}{2} \end{cases} \vee \quad z_3 = -6$ eco. niet relevant

Hessiaan $> 0 \rightarrow$ min

Antw: $q = z_2 + z_3 = 13{,}5$ ton

3 $\frac{D}{q} = \frac{135}{13{,}5} = 10$

4 z_2 = max. voorraad = 7,5 ton

z_3 : max. tekort = 6

5 1 jaar = 360 dagen

1 cyclus = 36 dagen = T

$\tau_2 + \tau_3 = 36$

$\frac{z_3}{z_2 + z_3} = \frac{\tau_3}{\tau_2 + \tau_3}$

$\Leftrightarrow \tau_3 = \frac{6 \cdot 36}{13{,}5} = 16$ dagen

15 $D = 1350$ $\quad q = z_2 + z_3$

$C_3 = 25$

$C_1 = 75$ $\quad \frac{\tau_2}{z_2} = \frac{\tau_3}{z_3} = \frac{\tau_2 + \tau_3}{z_2 + z_3}$

$C_2 = 20$

1 TK = bestelkosten + beheerskosten + voorraadtekortkosten
1 2 3

1) $C_1 \cdot \frac{D}{q} = \frac{C_1 D}{z_2 + z_3} = \frac{75 \cdot 1350}{z_2 + z_3} = \frac{101250}{z_2 + z_3}$

2) $\frac{1}{2} C_2 \frac{z_2^2}{z_2 + z_3} = \frac{10 z_2^2}{z_2 + z_3}$

3) $\frac{1}{2} C_3 \frac{z_3^2}{z_2 + z_3} = \frac{12{,}5 \, z_3^2}{z_2 + z_3}$

$K(z_2, z_3) = \frac{101250}{z_2 + z_3} + \frac{10 z_2^2}{z_2 + z_3} + \frac{12{,}5 z_3^2}{z_2 + z_3}$

2 $(z_2 + z_3) K(z_2, z_3) = 101250 + 10 z_2^2 + 12{,}5 \, z_3^2$

$$\begin{cases} 20 z_2 - K(z_2, z_3) = 0 & (1) \\ 25 z_3 - K(z_2, z_3) = 0 & (2) \end{cases}$$

$$\begin{array}{ll} (1) - (2) & \begin{cases} 20 z_2 = 25 \cdot z_3 & (3) \\ 20 z_2 + 25 z_3 - 2K(z_2, z_3) = 0 & (4) \end{cases} \\ (1) + (2) & \end{array}$$

(3) $\Leftrightarrow z_2 = \frac{5}{4} z_3$ (5)

(4) $(20 z_2 + 25 z_3)(z_2 + z_3) - 20 z_2^2 - 25 \cdot z_3^2 = 2 \cdot 101250$

$\Leftrightarrow 25 z_2 z_3 + 20 z_2 z_3 = 202500$

$\Leftrightarrow z_2 z_3 = 4500$ (6)

(6) in (5) $\begin{cases} z_3^* = 60 \\ z_1^* = 75 \end{cases}$

Antw $q = z_2 + z_3 = 135$

3 $T = 360$ $\quad \frac{D}{q} = \frac{1350}{135} = 10$

Antw: 10 cycli van 36 dagen

17 $S(x,y) = \frac{240x}{25+3x} + \frac{150y}{10+y}$

1 $W(x,y) = S(x,y) - 0{,}8\,S(x,y) - x - y$

$= \frac{48x}{25+3x} + \frac{30y}{10+y} - x - y$

2 $x + y - 15 = 0$ (nevenVW)

$\mathcal{L}(x,y,\lambda) = \frac{48x \overset{+400-400}{}}{25+3x} + \frac{30y \overset{+300-300}{}}{10+y} - x - y + \lambda(x+y-15) =$

$= 16 - \frac{400}{3x+25} + 30 \; \frac{-300}{y+10} - x - y + \lambda(x+y-15)$

$= 46 - \frac{400}{3x+25} - \frac{300}{y+10} - x - y + \lambda(x+y-15)$

$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = \frac{400 \cdot 3}{(3x+25)^2} - 1 + \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = \frac{300}{(y+10)^2} - 1 + \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 15 = 0 \quad (1) \end{cases}$

$\Rightarrow \frac{\partial^2 \mathcal{L}}{\partial x^2} = -1200 \cdot 6(3x+25) = -288\,000$

$\Rightarrow -300 \cdot 2(y+10) = -12000$

$1200(y+10)^2 = 300(3x+25)^2$

$\Leftrightarrow 4(y+10)^2 = (3x+25)^2$

$\Leftrightarrow \pm 2(y+10) = 3x+25$

$\Leftrightarrow 3x + 25 = -2y - 20 \quad \vee \quad 3x + 25 = 2y + 20$

$\Leftrightarrow x, y > 0 \quad$ / $\quad y = \frac{3x+5}{2} \quad (2)$

(2) in (1): $x + \frac{3x}{2} + \frac{5}{2} - 15 = 0$

$\Leftrightarrow \frac{5x}{2} = \frac{25}{2}$

$\Leftrightarrow \begin{cases} x = 5 \\ y = 10 \end{cases}$

$H^* = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -288000 & 0 \\ 1 & 0 & -12000 \end{vmatrix} = (-1)(-120000) + 1(288000) > 0 \rightarrow \text{MAX}$

Antw: radio: 50 000 $\quad W(5,10) = 60\,000$
reclame: 100 000

19 $q = 10 - \frac{1}{q_A} - \frac{1}{q_B}$ $\quad p_A = 1 \quad p_B = 4$

$W: p\,q - q_A \cdot 1 - q_B \cdot 4$

$\Leftrightarrow W: p\left(10 - \frac{1}{q_A} - \frac{1}{q_B}\right) - q_A - 4q_B = 72$

$$\begin{cases} \frac{\partial W}{\partial q_A} = \frac{+p}{q_A^2} - 1 = 0 \\ \frac{\partial W}{\partial q_B} = \frac{p}{q_B^2} - 4 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} p = q_A^2 \\ p = 4q_B^2 \end{cases} \Leftrightarrow \begin{cases} q_A = \sqrt{p} \\ q_B = \frac{\sqrt{p}}{2} \end{cases}$$

$W: p\left(10 - \frac{1}{\sqrt{p}} - \frac{2}{\sqrt{p}}\right) - \sqrt{p} - 2\sqrt{p} = 72$

$\Leftrightarrow 10p - \sqrt{p} - 2\sqrt{p} - \sqrt{p} - 2\sqrt{p} = 72$

$\Leftrightarrow 10p - 6\sqrt{p} - 72 = 0$

Stel $p = t^2$

$\Leftrightarrow 10t^2 - 6t - 72 = 0$

$\Leftrightarrow 5t^2 - 3t - 36 = 0$

$D = 9 + 720 = 729 \quad \sqrt{D} = 27$

$t_1 = \frac{3+27}{10} = 3$

$t_2 = \frac{3-27}{10} = - \ldots$

$\Rightarrow p = t^2 = 3^3 = 9$ ~~eenheden~~ verkoopprijs

20 $f(x,y) = \dfrac{xy}{x^2+y^2+1}$ $\quad G(x,y) = x^2+y^2-b$

1 $\mathcal{L}(x,y,\lambda) = \dfrac{xy}{x^2+y^2+1} + \lambda(x^2+y^2-b)$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{(x^2+y^2+1)y - xy(2x)}{(x^2+y^2+1)^2} + 2\lambda x = 0 \quad \cdot y \\ \dfrac{\partial \mathcal{L}}{\partial y} = \dfrac{(x^2+y^2+1)x - xy\,2y}{(x^2+y^2+1)^2} + 2\lambda y = 0 \quad \cdot x \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = x^2+y^2-b = 0 \quad (1) \end{cases}$$

$(x^2+y^2+1)y^2 - 2x^2y^2 = (x^2+y^2+1).x^2 - 2x^2y^2 = 0$

$\Leftrightarrow y^2 = x^2 \Leftrightarrow y = x \vee y = -x \vee -y = x \vee -y = -x$ (2)

(2) in (1)

$x^2 + x^2 - b = 0 \Leftrightarrow 2x^2 = b \Leftrightarrow x = \dfrac{\sqrt{b}}{\sqrt{2}} = \dfrac{\sqrt{2b}}{2} \vee x = -\dfrac{\sqrt{2b}}{2}$

Antw: $P_1\left(\dfrac{\sqrt{2b}}{2}, \dfrac{\sqrt{2b}}{2}\right), P_2\left(-\dfrac{\sqrt{2b}}{2}, \dfrac{\sqrt{2b}}{2}\right), P_3\left(\dfrac{\sqrt{2b}}{2}, -\dfrac{\sqrt{2b}}{2}\right), P_4\left(-\dfrac{\sqrt{2b}}{2}, -\dfrac{\sqrt{2b}}{2}\right)$

2 $\cancel{\dfrac{\partial \mathcal{L}}{\partial x}(2,a) = \dfrac{(5+a^2)a - 8a}{(5+a)^2} + 4\lambda = 0}$

$\cancel{\dfrac{\partial \mathcal{L}}{\partial y}(2,a) =}$

a) $P(2,a)$: $2 = \dfrac{\sqrt{2b}}{2} \Leftrightarrow b = 8$

$a = \dfrac{\sqrt{16}}{2} = -2 \vee a = 2$

b) $G(x,y) = x^2+y^2-8 = 0$

$\Leftrightarrow x^2+y^2 = 8$

→ cirkel $M(0,0)$ en straal $\sqrt{8}$

c) extremum: $G(x,y)$ $f(x,y)$ raakt

→ raakpunten cirkel met niveaulijnen

d) $P(2,a) \Rightarrow b = 8$ en $a = 2 \vee a = -2$

$P_1(2,2)$ $\quad$ $k\mathbb{P} \Rightarrow k+8$ $\quad$ $f(x,0)$: x^2-8

$P_3(2,-2)$

1	9
2	10
3	11

$H^* = \begin{vmatrix} 0 & 2x & 2y \\ 2x & & \\ 2y & & \end{vmatrix}$

$\frac{\partial^2 L}{\partial x^2} = (x^2+y^2+1)^2 \cdot (2xy - 4xy) - [(x^2+y^2+1)y - 2x^2y] \, 2(x$

a $f(x,y) = a^2x^2 + (6a-8)y^2 - 16x + 32y - 10 \qquad x \leq 10$

1 $\frac{\partial f}{\partial x} = 2a^2x - 16 = 0 \iff x = \frac{8}{a^2}$

$\frac{\partial f}{\partial y} = 2(6a-8)y + 32 = 0 \iff y = \frac{-16}{6a-8}$

2 $H^* = \begin{vmatrix} 2a^2 & 0 \\ 0 & 12a-16 \end{vmatrix}$ $\quad \det H_1 = 2a^2 > 0 \qquad \det H_2 = 24a^3 - 32a^2 = 4a^2(6a-8)$

3 $4a^2(6a-8) = 0$

$\iff a = 0 \vee a = \frac{4}{3}$

		0		4/3	
H_1	+		+		+
H_2	−		−		+
	zadel		zadel		min

3 $\mathcal{L}(x,y,\lambda) = x^2 - 2y^2 - 16x + 32y - 10 + \lambda(x - 3 - (y-8)^2)$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x - 16 + \lambda = 0 \rightarrow x = 8 \\ \frac{\partial L}{\partial y} = -4y + 32 - 2\lambda(y-8) = 0 \quad -4(y-8) = 2\lambda(y-8) \rightarrow \lambda = -2 \\ \frac{\partial L}{\partial \lambda} = x - 3 - (y-8)^2 = 0 \end{cases}$$

22 1 (a) $\frac{y^2-1}{x^2+1} > 0 \quad y^2-1 > 0 \rightarrow y^2 > 1$
$y > 1$
$y < -1$

def: $]-\infty, -1[\cup]1, +\infty[$

(b) $\mathcal{L}(x,y,\lambda) = \ln \frac{y^2-1}{x^2+1} + \lambda(x^2-y^2+a)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = \left(\frac{x^2+1}{y^2-1}\right) \frac{-2x(y^2-1)}{(x^2+1)^2} + 2\lambda x = 0 \quad \cdot y \\ \frac{\partial \mathcal{L}}{\partial y} = \left(\frac{x^2+1}{y^2-1}\right) \frac{2y(x^2+1)}{(x^2+1)^2} - 2\lambda y = 0 \quad \cdot x \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x^2-y^2+a = 0 \end{cases}$$

$\frac{2xy}{x^2+1} = \frac{2xy}{y^2-1} \Leftrightarrow \frac{1}{x^2+1} = \frac{1}{y^2-1}$

$\Leftrightarrow 2xy(x^2+1) = (y^2-1)\,2xy$

$\Leftrightarrow$ ~~$\begin{cases} x=0 \\ y=\sqrt{a} \vee -\sqrt{a} \end{cases}$ $(0,\sqrt{a}), (0,-\sqrt{a})$ $G(x,y)=0$~~

$\Leftrightarrow 2xy(x^2+1) - 2xy(y^2-1) = 0$

$\Leftrightarrow 2xy(x^2-y^2+2) = 0$

$\Leftrightarrow x=0 \vee y=0 \vee x^2-y^2+2=0$

niet mog wegens def(f)

$y=-\sqrt{a} \vee y=\sqrt{a} \quad \Rightarrow a=2$

$(0,\sqrt{2}) ; (0,-\sqrt{2})$

(c) $a=2$

2 (d) $(0,3) ; (0,-3)$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{-2x}{x^2+1} + 2\lambda x \qquad \frac{\partial^2 \mathcal{L}}{\partial x^2} = \frac{-2(x^2+1)+2x\,2x}{(x^2+1)^2} + 2\lambda$

$= \frac{-2x^2-2}{(x^2+1)^2} + 2\lambda$

$\frac{\partial \mathcal{L}}{\partial y} = \frac{2y}{y^2-1} - 2\lambda y \qquad = \frac{-2(x^2+1)}{(x^2+1)^2} + 2\lambda$

$\frac{\partial^2 \mathcal{L}}{\partial y^2} = \frac{2(y^2-1)-4y^2}{(y^2-1)^2} - 2\lambda \qquad = \frac{-2}{x^2+1} + 2\lambda$

$= \frac{-2(y^2+1)}{(y^2-1)^2} - 2\lambda \qquad \frac{\partial \mathcal{L}}{\partial y \partial x} = 0$

SP $(0, 3)$

$$\begin{cases} x = 0 \\ y = 3 \\ \frac{2y}{y^2-1} - 2y\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 3 \\ \frac{6}{8} - 6\lambda = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = 3 \\ \lambda = 1/8 \end{cases}$$

SP $(0, -3)$

$$\begin{cases} x = 0 \\ y = -3 \\ \frac{-6}{8} + 6\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -3 \\ \lambda = 1/8 \end{cases}$$

$$H^*(0,3) = \begin{vmatrix} 0 & 0 & -6 \\ 0 & -7/4 & 0 \\ -6 & 0 & -9/16 \end{vmatrix} = -6 \cdot -(\ldots) > 0 \quad \max$$

$$H^*(0,-3) = \begin{vmatrix} 0 & 0 & 6 \\ 0 & -7/4 & 0 \\ 6 & 0 & -9/16 \end{vmatrix} = 6 \cdot (+\ldots) > 0 \quad \max$$

~~3 d)~~

$$\begin{cases} \frac{\partial f}{\partial x} = -a e^{\cdots} + (1 - ax - by)\, e^{\cdots}\, (-2x) = 0 & (1) \\ \frac{\partial f}{\partial y} = -b e^{\cdots} + (1 - ax - by)\, e^{\cdots} = 0 & (2) \end{cases}$$

$\Leftrightarrow \cancel{e^{(x+y-x^2)} (-a - 2x(1 - ax - by))}$

(1) $e^{(x+y-x^2)} (-a - 2x(1 - ax - by)) = 0$

$$\begin{cases} \Leftrightarrow -a - 2x(1 - ax - by) = 0 \\ e^{\cdots} (-b + 1 - ax - by) = 0 \end{cases}$$ (2)

$$\Leftrightarrow \begin{cases} -a + 1 + \frac{a}{2} - by = 0 \\ -b + 1 + \frac{1}{2}a - by = 0 \end{cases}$$

$\Leftrightarrow \cancel{-\frac{3}{2}a - b + 1 + \frac{1}{2}a = 0}$

$\Leftrightarrow \cancel{a - b = 0}$

23 : (a) $\begin{cases} \dfrac{\partial f}{\partial x} = -a e^{\cdots} + (1-ax-by)(-2x) e^{\cdots} = 0 \\ \\ \dfrac{\partial f}{\partial y} = -b e^{\cdots} + (1-ax-by) \cdot e^{\cdots} = 0 \end{cases}$

$\begin{cases} \dfrac{\partial f}{\partial x} = e^{\cdots} (-a - 2x(1-ax-by)) = 0 \\ \dfrac{\partial f}{\partial y} = e^{\cdots} (-b + 1 - ax - by) = 0 \end{cases}$

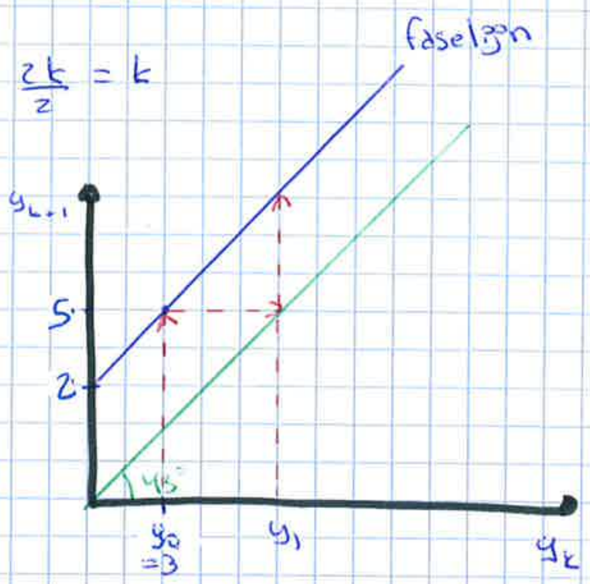
$$-a - 2x(1-ax-by) = -b + 1 - ax - by$$

$\Leftrightarrow \ -a$

Differentievergelijkingen

1 1 $s - s = 0$

2 $\frac{(k+1)k}{2} + C - \left(\frac{k(k-1)}{2} + C\right) = \frac{k(k+1-k+1)}{2} = \frac{2k}{2} = k$



2 1 Stap 1 $s - 1 = 0 \Leftrightarrow s_0 = 1$

Stap 2 $y_g(k) = C.1^k = C$

Stap 3 $y_p(k) = Ck$

+4

$C(k+1) - Ck = 2$

$\Leftrightarrow C = 2$

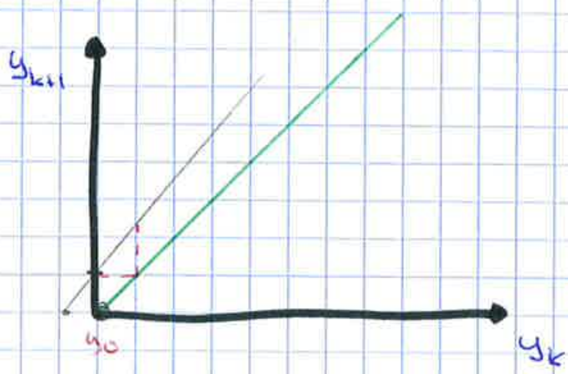
Stap 5: AO: $y_k = C + 2k$

Stabiliteit: $\lim_{k \to +\infty} y_k = +\infty$ divergent, onstabiel

Stap 6: $y_0 = 3$

$\Leftrightarrow C = 3$

$\Rightarrow$ A.O: $3 + 2k$



2 1) $s - 3 = 0 \Leftrightarrow s = 3$

2) $y_g(p) = C.3^k$

3) $y_p(k) = C \Leftrightarrow C - 3C = 1$

$\Leftrightarrow C = -\frac{1}{2}$

4) $y_k = C.3^k - \frac{1}{2}$

Stab: $\lim_{k \to +\infty} y_k = +\infty$, $\to$ divergent, onstabiel

6) P.O. $y_0 = 0 \Leftrightarrow 0 = C - \frac{1}{2} \Leftrightarrow C = \frac{1}{2}$

$y_k = \frac{1}{2} 3^k - \frac{1}{2}$

3 1) $s + 1 = 0 \Leftrightarrow s = -1 \Rightarrow y_g(k) = C(-1)^k$

3) $y_p(k) = C \Rightarrow C + C = +2 \Leftrightarrow C = +1$

5) $y_k = C(-1)^k + 1$

$\lim_{k \to +\infty} y_k = \pm\infty$ niet-convergent, niet stabiel

6) $y_0 = 1 \Leftrightarrow 1 = C + 1 \Leftrightarrow C = 0$

4 $3y_{k+1} - 2y_k = 3$

1) $3s - 2 = 0 \Leftrightarrow s = \frac{2}{3}$

2) $y_g(k) = C\left(\frac{2}{3}\right)^k$

3) $y_p(k) = c$

$3c - 2c = 3 \Leftrightarrow c = 3$

5) $y_k = C\left(\frac{2}{3}\right)^k + 3$

$\lim_{k\to+\infty} y_k = 3$ convergent, stabiel

6) $y_0 = 0 \Leftrightarrow 0 = C + 3 \Leftrightarrow C = -3$

6 1) $s + 3 = 0 \Leftrightarrow s = -3 \Rightarrow y_g(p) = C(-3)^k$

3) $y_p(k) = c \Rightarrow c + 3c = 0 \Leftrightarrow c = 0$

5) $y_k = C(-3)^k$

$\lim_{k\to+\infty} y_k = \pm\infty$ niet-convegent, onstabiel

6) $y_0 = 1 \Rightarrow C = 1$

3 1) $y_1 = \frac{y_0}{1+y_0} \neq 0 \rightsquigarrow y_k \neq 0$

2 $\frac{1}{f_{k+1}} = \frac{1/f_k}{1+\frac{1}{f_k}} \Leftrightarrow \frac{1}{f_{k+1}} = \frac{1}{f_k} \cdot \frac{f_k}{f_k+1}$

$\Leftrightarrow f_k + 1 = f_{k+1}$

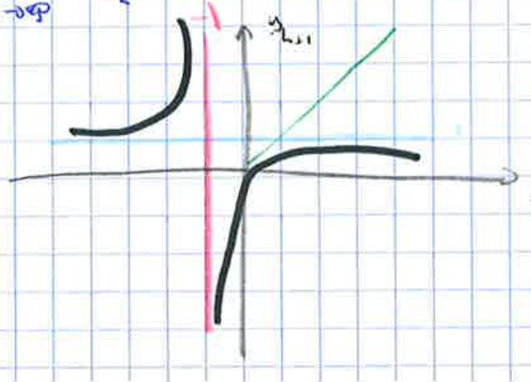
3 $f_{k+1} - f_k = 1$

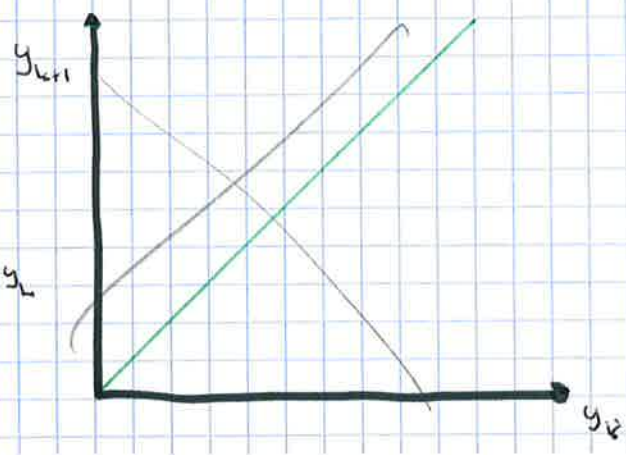
1) $s = 1 \Rightarrow y_g(k) = C$

3) $y_p(k) = ck \Rightarrow c(k+1) - c = 1$
$\Leftrightarrow c = k$

5) $f_k = C + k \Rightarrow y_k = \frac{1}{C+k}$

4 $\lim_{k\to+\infty} y_k = 0$ stabiel





4 1 $y_{k+2} - 3y_{k+1} + 2y_k = 0 \quad , \quad y_0 = 0 \, , y_1 = 1$

1) $s^2 - 3s + 2 = 0$

$D = 9 - 8 = 1$

$s_1 = \frac{3+1}{2} = 2$

$s_2 = \frac{3-1}{2} = 1$

2) $y_g(k) = C_1 \cdot 2^k + C_2 \cdot 1^k$

→ homogeen: $y_k = C_1 2^k + C_2$

$\lim_{k \to \infty} y_k = +\infty$ divergent, niet stabiel

6) $\begin{cases} y_0 = 0 \Leftrightarrow 0 = C_1 + C_2 \\ y_1 = 1 \Leftrightarrow 1 = 2C_1 + C_2 \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$

$y_k = 2^k - 1$

2 1) $s^2 - 2s + 1 = 0$

$D = 4 - 4 = 0$

$s = \frac{2}{2} = 1$ en $s = k \cdot 1^k$

$\Rightarrow y_g(k) = C_1 + C_2 k$

→ homogeen: $y_k = C_1 + C_2 k$

$\lim_{k \to +\infty} y_k = +\infty$ onstabiel

6) $\begin{cases} y_0 = 1 \Leftrightarrow 1 = C_1 \\ y_1 = 3 \Leftrightarrow 3 = C_1 + C_2 \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = 2 \end{cases}$

3 1) $s^2 + 1 = 0 \qquad D = 0 - 4 = -4 < 0$

$p = \frac{-a_1}{2} = 0 \qquad q = \frac{\sqrt{|D|}}{2} = 1 \qquad r = \sqrt{p^2 + q^2} = 1$

$\arctan \frac{q}{p} = \theta \Leftrightarrow \theta = \arctan \frac{1}{0}$

$\Leftrightarrow \theta = \frac{\pi}{2}$

5) $y_k = r^k (C_1 \cos \theta k + C_2 \sin \theta k) = C_1 \cos \frac{\pi}{2} k + C_2 \sin \frac{\pi}{2} k$

→ $r = 1$ oscileert met constante amplitude

→ niet stabiel

6) $\begin{cases} y_0 = 1 \Leftrightarrow 1 = C_1 \cos 0 + C_2 \sin 0 \\ y_1 = 0 \Leftrightarrow 0 = C_1 \cos \frac{\pi}{2} + C_2 \sin \frac{\pi}{2} \end{cases}$

$\Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$

$y_p(k) = \cos \frac{\pi}{2} k$

4 1) $8s^2 - 6s + 1 = 0$

$D = 36 - 32 = 4$

$s_1 = \frac{6-2}{16} = \frac{1}{4}$

$s_2 = \frac{6+2}{16} = \frac{1}{2}$

$\Rightarrow y_k = C_1\left(\frac{1}{4}\right)^k + C_2\left(\frac{1}{2}\right)^k$

$\lim_{k \to +\infty} y_k = 0$ stabiel

6) $\begin{cases} y_0 = 1 \Leftrightarrow 1 = C_1 + C_2 \\ y_1 = 0 \Leftrightarrow 0 = \frac{1}{4} C_1 + \frac{1}{2} C_2 \end{cases}$ $\begin{cases} C_1 = 1 - C_2 & (1) \\ 0 = \frac{1}{4} - \frac{1}{4} C_2 + \frac{1}{2} C_2 & (2) \end{cases}$

(2) $\frac{1}{4} C_2 = -\frac{1}{4} \Leftrightarrow C_2 = -1$ (3)

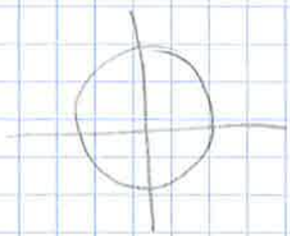
(3) in (1) $C_1 = 1 + 1 = 2$

5 1) $s^2 + 2s + 2 = 0$

$D = 4 - 8 = -4 < 0$

$p = \frac{-a_1}{2} = -1 \quad q = \frac{\sqrt{|D|}}{2} = 1$

$r = \sqrt{p^2 + q^2} = \sqrt{2}$



$\theta = \arctan \frac{q}{p} \Leftrightarrow \theta = \arctan -1$

$\Leftrightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

8) $y_k = \sqrt{2}^k \left(C_1 \cos k\frac{3\pi}{4} + C_2 \sin k\frac{3\pi}{4}\right)$

$r > 1 \rightarrow$ divergent, onstabiel

6) $\begin{cases} y_0 = 1 \Leftrightarrow 1 = C_1 \cos 0 + C_2 \sin 0 \Leftrightarrow C_1 = 1 & (1) \\ y_1 = 1 \Leftrightarrow 1 = \sqrt{2} C_1 \cos \frac{3\pi}{4} + \sqrt{2} C_2 \sin \frac{3\pi}{4} & (2) \end{cases}$

(2) $1 = \sqrt{2} \cdot \frac{-\sqrt{2}}{2} + \sqrt{2} \cdot C_2 \left(\frac{\sqrt{2}}{2}\right) \Leftrightarrow 1 = -1 + \frac{1}{2} C_2 \Rightarrow C_2 = 4$?

7 1) $s^2 - 3s + 3 = 0$

$D = 9 - 12 = -3$

$p = \frac{3}{2} \quad q = \frac{\sqrt{3}}{2} \qquad r = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

$\theta = \tan\frac{q}{p} \Leftrightarrow \theta = \arctan\frac{\sqrt{3}}{3} = \frac{\pi}{6}$

2) $y_g(k) = \sqrt{3}^k \left(C_1 \cos\frac{\pi}{6}k + C_2 \sin\frac{\pi}{6}k\right)$

3) $y_p(k) = C \Leftrightarrow C - 3C + 3C = 5$
$\Leftrightarrow C = 5$

5) $y_k = \sqrt{3}^k \left(C_1 \cos\frac{\pi}{6}k + C_2 \sin\frac{\pi}{6}k\right) + 5$

$\lim_{k\to+\infty} y_k = +\infty$ <u>onstabiel</u>

6) $\begin{cases} y_0 = 5 \Leftrightarrow 5 = C_1 \cos 0 + C_2 \sin 0 + 5 & (1) \\ y_1 = 8 \Leftrightarrow 8 = \sqrt{3}\left(C_1 \cos\frac{\pi}{6} + C_2 \sin\frac{\pi}{6}\right) + 5 & (2) \end{cases} \Leftrightarrow \begin{cases} C_1 = 0 \\ \end{cases}$

(2) $8 = \sqrt{3}\, C_2 \cdot \frac{1}{2} + 5 \Leftrightarrow \frac{6}{\sqrt{3}} = C_2 \Leftrightarrow C_2 = 2\sqrt{3}$

$y_k = \sqrt{3}^k \cdot 2\sqrt{3} \sin\left(\frac{\pi}{6}k\right) + 5$

8 1) $9s^2 - 6s + 1 = 0$

$D = 36 - 36 = 0$

$s_1 = \frac{6}{18} = \frac{1}{3} \qquad s_2 = k\,\frac{1}{3}$

$\Rightarrow y_g(k) = C_1\left(\frac{1}{3}\right)^k + C_2 k \cdot \left(\frac{1}{3}\right)^k$

3) $y_p(k) = C \Leftrightarrow 9C - 6C + C = 16$
$\Leftrightarrow C = 4$

5) $y_k = C_1\left(\frac{1}{3}\right)^k + C_2 k \left(\frac{1}{3}\right)^k + 4$

$\rightarrow \lim_{k\to+\infty} y_k = 4 \rightarrow$ <u>stabiel</u>

6) $\begin{cases} y_0 = 0 \Leftrightarrow C_1 + 4 = 0 \\ y_1 = 3 \Leftrightarrow C_1 \cdot \frac{1}{3} + C_2 \cdot \frac{1}{3} + 4 = 3 \end{cases} \Leftrightarrow \begin{cases} C_1 = -4 & (1) \\ -\frac{4}{3} + C_2 \frac{1}{3} + 4 = 3 & (2) \end{cases}$

(2) $C_2 \frac{1}{3} = \frac{1}{3} \Leftrightarrow C_2 = 1$

12 1) $3s^2 + 5s + 2 = 0$

$D = 25 - 24 = 1$

$s_1 = \frac{-5+1}{6} = -\frac{2}{3}$ $\quad s_2 = \frac{-5-1}{6} = -1$

$\Rightarrow y_g(k) = C_1(-1)^k + C_2\left(\frac{-2}{3}\right)^k$

3) $y_p(k) = c \Leftrightarrow 3c + 5c + 2c = 4$

$\Leftrightarrow c = \frac{2}{5}$

5) $y_k = C_1(-1)^k + C_2\left(\frac{-2}{3}\right)^k + \frac{2}{5}$

$\lim_{k\to+\infty} y_k = \pm\infty$ niet convergent, <u>niet stabiel</u>

fout $\rightarrow$ 6) $\begin{cases} y_0 = 1 \Leftrightarrow 1 = C_1 + C_2 + \frac{2}{5} \\ y_1 = 1 \Leftrightarrow 1 = -C_1 * -\frac{2}{3}C_2 + \frac{2}{5} \end{cases}$ $\Leftrightarrow \begin{cases} C_1 = \frac{3}{5} - C_2 & (1) \\ 1 = -\frac{3}{5} + C_2 - \frac{2}{5}C_2 + \frac{2}{5} & (2) \end{cases}$

(2) $1 = \frac{-1}{5} + \frac{3}{5}C_2$ $\quad$ (3) in (1) $C_1 = \frac{3}{5} - 2$

$\Leftrightarrow 5 = -1 + 3C_2$ $\quad = -7/5$

$\Leftrightarrow 6 = 3C_2$

$\Leftrightarrow C_2 = 2$ (3)

5 1 1) $s^2 + -3s + 2 = 0$

$D = 9 - 8 = 1$

$s_1 = \frac{3+1}{2} = 2$ $\quad s_2 = \frac{3-1}{2} = 1$

$\Rightarrow y_g(k) = C_1 + 2^k C_2$

3) $y_p(k) = A \cdot 3^k$

$\Rightarrow y_p(k) = A(9.3^k - 9.3^k + 2.3^k) = 3^k$

$\Leftrightarrow 2A = 1$

$\Leftrightarrow A = 1/2$

5) $y_k = C_1 + 2^k C_2 + \frac{1}{2} \cdot 3^k$

2 1) $s^2 - 3s + 2 = 0$

$D = 9 - 8 = 1$

$s_1 = \frac{3+1}{2} = 2 \qquad s_2 = \frac{3-1}{2} = 1$

$\Rightarrow y_g(k) = C_1 + C_2 2^k$

3) $y_p(k) = A.k.2^k$

$\Leftrightarrow A\left[(k+2)4 \cdot 2^k - 3(k+1).2.2^k + 2k2^k\right] = 2^k$

$\Leftrightarrow A(4k + 8 - 6k - 6 + 2k) = 1$

$\Leftrightarrow 4Ak - 6AK + 2Ak + 8A - 6A = 1$

$\Leftrightarrow \begin{cases} 4A - 6A + 2A = 0 \\ 8A - 6A = 1 \end{cases} \Leftrightarrow A = 1/2$

5) $y_k = C_1 + C_2.2^k + \frac{1}{2}k.2^k = C_1 + 2^k C_2 + k.2^{k-1}$

4 1) $s^2 - 6s + 1 = 0$

$D = 36 - 4 = 32 \qquad \sqrt{D} = 4\sqrt{2}$

$s_1 = \frac{6 + 4\sqrt{2}}{2} = 3 + 2\sqrt{2}$

$s_2 = \frac{6 - 4\sqrt{2}}{2} = 3 - 2\sqrt{2}$

$\Rightarrow y_g(k) = C_1(3 + 2\sqrt{2})^k + C_2(3 - 2\sqrt{2})^k$

3) $y_p(k) = A\cos(k\alpha) + B\sin(k\alpha)$

$= A\cos(k\frac{\pi}{2}) + B\sin(k\frac{\pi}{2})$

4) $A\cos\left[(k+2)\frac{\pi}{2}\right] + B\sin\left[(k+2)\frac{\pi}{2}\right] - 6A\cos\left[(k+1)\frac{\pi}{2}\right] - 6B\sin\left[(k+1)\frac{\pi}{2}\right]$

$+ A\cos k\frac{\pi}{2} + B\sin k\frac{\pi}{2} = 5\sin\frac{\pi}{2}k$

$\cos(\frac{\pi}{2}k + \pi) = \cos(\frac{\pi}{2}k)\cos\pi - \sin(\frac{\pi}{2}k)\sin\pi = -\cos(\frac{\pi}{2}k)$

$\cos(\frac{\pi}{2}k + \frac{\pi}{2}) = \cos(\frac{\pi}{2}k)\cos\frac{\pi}{2} - \sin(\frac{\pi}{2}k)\sin(\frac{\pi}{2}) = -\sin\frac{\pi}{2}k$

$\sin(\frac{\pi}{2}k + \pi) = \cos(\frac{\pi k}{2})\sin\pi + \cos\pi.\sin\frac{\pi}{2}k = -\sin\frac{\pi}{2}k$

$\sin(\frac{\pi}{2}k + \frac{\pi}{2}) = \cos(\frac{\pi}{2}k)\sin\frac{\pi}{2} + \sin\frac{\pi}{2}k.\cos\frac{\pi}{2} = \cos\frac{\pi}{2}k$

$\Rightarrow -A\cos(\frac{\pi}{2}k) - B\sin(\frac{\pi}{2}k) + 6A\sin(\frac{\pi}{2}k) - 6B\cos\frac{\pi}{2}k + A\cos(k\frac{\pi}{2})$

$+ B\sin(\frac{k\pi}{2}) = 5\sin(\frac{\pi}{2}k)$

$\Leftrightarrow (-A - 6B + A)\cos\left(\frac{k\pi}{2}\right) + (-B + 6A + B)\sin\left(\frac{k\pi}{2}\right) = 5\sin\left(\frac{k\pi}{2}\right)$

$\Leftrightarrow \begin{cases} -6B = 0 \\ 6A = 5 \end{cases} \Leftrightarrow \begin{cases} B = 0 \\ A = 5/6 \end{cases}$

$\Rightarrow y_p(k) = \frac{5}{6}\sin\left(\frac{k\pi}{2}\right)$

5) $y_p = C_1(3+2\sqrt{2})^k + C_2(3-2\sqrt{2})^k + \frac{5}{6}\sin\left(k\frac{\pi}{2}\right)$

5 1) $s^2 - 3s + 2 = 0$

$D = 9 - 8 = 1$

$s_1 = \frac{3+1}{2} = 2 \qquad s_2 = \frac{3-1}{2} = 1$

$\Rightarrow y_g(k) = C_1 + 2^k C_2$

3) $y_p(k) = k \cdot 2^k(3k^A + B)$

4) $(k+2) \cdot 4 \cdot 2^k[3(k+2)+1] - 3(k+1) \cdot 2 \cdot 2^k[3(k+1)+1] + 2k \cdot 2^k(3k+1)$

$= 2^k(3k+1)$

$\Leftrightarrow (4k+8)(3k+7) - (6k+6)(3k+4) + 2k(3k+1) = 3k+1$

$\Leftrightarrow 12k^2 + 28k + 24k + 56 - 18k^2 - 24k - 18k - 24 + 6k^2 + 2k = 3k+1$

$\Leftrightarrow 12 - 18 + 6$

$(k+2) \cdot 4 \cdot 2^k(3A(k+2)+B) - 3(k+1) \cdot 2 \cdot 2^k(3A(k+1)+B)$

$+ 2k \cdot 2^k(3Ak+B) = 2^k(3k+1)$

$\Leftrightarrow (4k+8)(3Ak+6A+B) - (6k+6)(3Ak+3A+B) + 2k(3Ak+B)$

$= 3k+1$

$\Leftrightarrow 12Ak^2 + 24Ak + 4Bk + 24Ak + 48A + 8B - 18Ak^2 - 18Ak - 6kB$

$- 18Ak - 18A - 6B + 6Ak^2 + 2kB = 3k+1$

$\Leftrightarrow \begin{cases} 12A - 18A + 6A = 0 \\ 24A + 4B + 24A - 18A - 6B - 18A + 2B = 3 \\ 48A + 8B - 18A - 6B = 1 \end{cases}$

$\Leftrightarrow \begin{cases} 12A = 3 \\ 30A + 2B = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1/4 \\ B = 1/8 \end{cases}$ → klopt niet kom! ten volle niet

6. 1) $s^2 - 2s + 1 = 0$

$D = 4 - 4 = 0$

$s_1 = \frac{2}{2} = 1 \qquad s_2 = k$

$\Rightarrow y_g(k) = C_1 + kC_2$

3) $y_p(k) = k^2(A3k + B4)$

4) $(k+2)^2[3A(k+2) + 4B] - 2(k+1)^2[3A(k+1) + 4B] + k^2(3Ak + 4B)$

$= 3k + 4$

$\Leftrightarrow (k^2 + 4k + 4)(3Ak + 6A + 4B) - (2k^2 + 4k + 2)(3Ak + 3A + 4B) +$

$k^2(3Ak + 4B) = 3k + 4$

$\Leftrightarrow 3\cancel{A}k^3 + 6\cancel{A}k^2 + 4\cancel{B}k^2 + 12\cancel{A}k^2 + 24Ak + 16kB + \cancel{12Ak} + 24A + \cancel{16B}$

$- 6\cancel{A}k^3 - 6\cancel{A}k^2 - 8\cancel{k^2}B - \cancel{12A}k^2 - \cancel{12}Ak - \cancel{16B} - 6Ak - 6A - 8B$

$+ 3\cancel{A}k^3 + 4\cancel{B}k^2 = 3k + 4$

$\Leftrightarrow \begin{cases} 24A + 16B - 6A = 3 \\ 24A - 6A - 8B = 4 \end{cases} \Leftrightarrow \begin{cases} 18A + 16B = 3 \\ 18A - 8B = 4 \end{cases} \Leftrightarrow \begin{cases} 18A = 3 - 16B & (1) \\ 3 - 16B - 8B = 4 & (2) \end{cases}$

$(2) \quad -24B = 7$

7 1) $s^2 + 1 = 0 \qquad D = -4 < 0$

$p = 0 \quad q = \frac{\sqrt{|D|}}{2} = 1 \qquad r = 1$

$\arctan \frac{1}{0} = \arctan \infty = \frac{\pi}{2}$

$\Rightarrow y_g(k) = C_1 \cos k\frac{\pi}{2} + C_2 \sin k\frac{\pi}{2}$

3) $y_p(k) = \left(A\cos\frac{\pi}{2}k + B\sin\frac{\pi}{2}k\right) \cdot k \qquad r = 1 \;\; \alpha \;\; \theta = \alpha$

4) $(k+2)\left[A\cos\left(\frac{\pi}{2}(k+2)\right) + B\sin\left(\frac{\pi}{2}(k+2)\right)\right] + k\left[A\cos\left(\frac{\pi}{2}k\right) + B\sin\left(\frac{\pi}{2}k\right)\right]$

$= \cos\frac{\pi}{2}k$

$\Leftrightarrow (k+2)\left(-A\cos\frac{\pi}{2}k - B\sin\frac{\pi}{2}k\right) + k\left(A\cos\left(\frac{\pi}{2}k\right) + B\sin\left(\frac{\pi}{2}k\right)\right) = \cos\frac{\pi}{2}k$

$\Leftrightarrow -A\cos\frac{\pi}{2}k - B\sin\frac{\pi}{2}k + A\cos\frac{\pi}{2}k + B\sin\frac{\pi}{2}k = 0$

$\begin{cases} -2A = 1 \\ -2B = 0 \end{cases} \qquad \begin{cases} A = -1/2 \\ B = 0 \end{cases}$

$\Rightarrow y_p = C_1\cos\frac{k\pi}{2} + C_2\sin\frac{k\pi}{2} - \frac{1}{2}\cos\frac{\pi}{2}k$

5 1) $s^2 - 3s + 2 = 0$

$D = 9 - 8 = 1$

$s_1 = \frac{3+1}{2} = 2 \qquad s_2 = \frac{3-1}{2} = 1$

$\Rightarrow y_g(k) = C_1 + 2^k C_2$

3) $y_p(k) = 2^k (Ak + B)\, k$

$\boxed{k=-2} : \begin{cases} -6(-A+B)(-1) + 2(-2A+B)(-2) = -5 \\ 4(2A+B)\,2 - 6(A+B) = 1 \end{cases}$ $\boxed{k=0}$

$\begin{cases} -6A + 6B + 8A - 4B = -5 \\ 16A + 8B - 6A - 6B = 1 \end{cases}$

$\Leftrightarrow \begin{cases} 2A + 2B = -5 & (1) \\ 10A + 2B = 1 & (2) \end{cases}$

(1) - (2) : $-8A = -6 \Leftrightarrow A = \frac{3}{4}$ (3)

(3) in (1) $\frac{3}{2} + 2B = -5 \Leftrightarrow B = -\frac{13}{4}$

5) $y_k = C_1 + 2^k C_2 + 2^k \left(\frac{3}{4}k^2 - \frac{13}{4}k\right)$

6 1) $s^2 - 2s + 1 = 0$

$D = 4 - 4 = 0$

$s_1 = \frac{2}{2} = 1 \quad \cup \quad s_1 = k$

$\Rightarrow y_g(k) = C_1 + kC_2$

3) $y_p(k) = k^2 (Ak + B)$

$\boxed{k=-2} : \begin{cases} -2\cdot(-1)^2(-A+B) + (-2)^2(-2A+B) = -2 \\ 2^2(2A+B) - 2\cdot 1^2 (A+B) = 4 \end{cases}$ $\boxed{k=0}$

$\begin{cases} 2A - 2B - 8A + 4B = -2 \\ 8A + 4B - 2A - 2B = 4 \end{cases}$

$\begin{cases} -6A + 2B = -2 & (1) \\ 6A + 2B = 4 & (2) \end{cases}$ (1) - (2) $-12A = -6 \Leftrightarrow A = 1/2$ (3)

(3) in (2) $3 + 2B = 4 \Leftrightarrow B = 1/2$

5) $y_k = C_1 + kC_2 + \frac{1}{2}(k^3 + k^2)$

6 $y_{k+2} + 2\alpha y_{k+1} + (\alpha - \frac{1}{4}) y_k = 0$

1) 1) $s^2 + 2\alpha s + (\alpha - \frac{1}{4}) = 0$

$$D = 4\alpha^2 - 4\alpha + 1 = 4\left(\alpha^2 - \alpha + \frac{1}{4}\right) = 4\left(\alpha - \frac{1}{2}\right)^2$$

$$s_1 = \frac{-2\alpha + 2(\alpha - \frac{1}{2})}{2} = \frac{-1}{2}$$

$$s_2 = \frac{-2\alpha - 2(\alpha - 1/2)}{2} = -2\alpha + \frac{1}{2}$$

$\Rightarrow \cancel{g}\ y_g(k) = y_k = \left(\frac{-1}{2}\right)^k C_1 + \left(-2\alpha + \frac{1}{2}\right)^k C_2$

$\alpha = \frac{1}{2}$ $\quad y_k = \left(\frac{-1}{2}\right)^k C_1 + \left(\frac{-1}{2}\right)^k C_2 k$

$C_0\ s_0 = \frac{1}{2} \quad s_1 = k \frac{1}{2}$

$\lim_{k \to +\infty} y_k = 0$ stabiel

$$\lim_{k \to +\infty} C_1 \left(\frac{-1}{2}\right)^k + \left(-2\alpha + \frac{1}{2}\right)^k C_2 = 0$$

$\Leftrightarrow \left|-2\alpha + \frac{1}{2}\right| < 1$

$\Leftrightarrow \begin{cases} -2\alpha + \frac{1}{2} < 1 \\ -2\alpha + \frac{1}{2} > -1 \end{cases} \vee \cancel{-2\alpha - \frac{1}{2}}$

$\Leftrightarrow \begin{cases} \alpha > -1/4 \\ \alpha < 3/4 \end{cases} \qquad \alpha \in \left]-1/4, 3/4\right[$

2 $y_{k+2} - \frac{1}{4} y_k = 0$

$\Rightarrow y_g(k) = \left(\frac{-1}{2}\right)^k C_1 + \cancel{\left(\frac{1}{2}\right)^k C_2} \left(\frac{1}{2}\right)^k C_2$

3) $y_p(k) = k \left(\frac{-1}{2}\right)^k$

Oefeningenexamens

1998-1999: R 1

1 $U(q_1, q_2) = q_1^2 + 5q_1q_2 + q_2^2$

1) $U(1,2) = 1 + 10 + 4 = 15 = q_1^2 + 5q_1q_2 + q_2^2$

$$\frac{\partial U}{\partial q_1} = 2q_1 + 5q_2 \Rightarrow 2 + 10 = 12$$

$$\frac{\partial U}{\partial q_2} = 5q_1 + 2q_2 \Rightarrow 5 + 4 = 9$$

rico raaklijn: $-\frac{12}{9} = -\frac{4}{3}$

$$t: q_2 - 2 = -\frac{4}{3}(q_1 - 1) \Leftrightarrow 3q_2 - 6 = -4q_1 + 4$$

$$\Leftrightarrow 3q_2 + 4q_1 = 10$$

2) $\mathcal{L}(q_1, q_2, \lambda) = q_1^2 + 5q_1q_2 + q_2^2 + \lambda(3q_1 + 2q_2 - 34)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial q_1} = 2q_1 + 5q_2 + 3\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial q_2} = 5q_1 + 2q_2 + 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 3q_1 + 2q_2 - 34 = 0 \end{cases} \qquad \begin{cases} q_1 = 4 \\ q_2 = 11 \\ \lambda = -21 \end{cases}$$

$$H^* = \begin{vmatrix} 0 & 3 & 2 \\ 3 & 2 & 5 \\ 2 & 5 & 2 \end{vmatrix} = 34 > 0 \Rightarrow \max$$

$U(4, 11) = 357$

3) $\mathcal{L}(q_1, q_2, \lambda) = q_1^2 + 5q_1q_2 + q_2^2 + \lambda(3q_1 + p_2q_2 - b)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial q_1} = 2q_1 + 5q_2 + 3\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial q_2} = 5q_1 + 2q_2 + p_2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 3q_1 + p_2q_2 - b = 0 \end{cases} \overset{(4,2)}{\Rightarrow} \begin{cases} 18 + 3\lambda = 0 \\ 24 + p_2\lambda = 0 \\ 12 + 2p_2 - b = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = -6 \\ p_2 = 4 \\ b = 20 \end{cases}$$

II $y_{k+2} + a y_{k+1} + \frac{1}{4} y_k = 0$

1) $s^2 + as + \frac{1}{4} = 0$ $\quad D = a^2 - 1 = 0$

$\Rightarrow a^2 - 1 < 0$

$\Leftrightarrow a < 1 \lor a < -1$

$\left(p = \frac{-a}{2} = 0 \quad q = \frac{\sqrt{|a^2-1|}}{2} \quad r = \right) \quad a \in \left]-1, 1\right[$

2) $y_{k+2} - 3y_{k+1} + \frac{1}{4} y_k = 0$

a) 1) $s^2 - 3s + \frac{1}{4} = 0$ $\quad D = 9 - 1 = 8 = 4.2$

$s_1 = \frac{3 + 2\sqrt{2}}{2} = \frac{3}{2} + \sqrt{2}$

$s_2 = \frac{3 - 2\sqrt{2}}{2} = \frac{3}{2} - \sqrt{2}$

$\Rightarrow y_g(k) = \left(\frac{3}{2} + \sqrt{2}\right)^k C_1 + \left(\frac{3}{2} - \sqrt{2}\right)^k C_2$

$\lim_{k \to \infty} y_k = +\infty \to$ onstabiel

b) $\begin{cases} y_0 = 0 \Leftrightarrow C_1 + C_2 = 0 \\ y_1 = 2\sqrt{2} \Leftrightarrow \left(\frac{3}{2} + \sqrt{2}\right) C_1 + \left(\frac{3}{2} - \sqrt{2}\right) C_2 = 2\sqrt{2} \end{cases}$

$\Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$

$y_p(k) = \left(\frac{3}{2} + \sqrt{2}\right)^k - \left(\frac{3}{2} + \sqrt{2}\right)^k$

c) $y_p(k) = \sqrt{2}^k \left(A \cos \frac{\pi}{4} k + B \sin \frac{\pi}{4} k\right)$

d) $y_p(k) = k(Ak + B)\left(\frac{3}{2} + \sqrt{2}\right)^k$

1998-1999: R2

I $U(q_1, q_2) = q_1^2 + 6q_1q_2 + 2q_2^2 + 5q_1$

1) $q_1^2 + 6q_1q_2 + 2q_2^2 + 5q_1 = 14$

$\frac{\partial U}{\partial q_1} = 2q_1 + 6q_2 + 5 \Rightarrow 13$

$\frac{\partial U}{\partial q_2} = 6q_1 + 4q_2 \Rightarrow 10$

rico: $-\frac{13}{10}$

$q_2 - 1 = -\frac{13}{10}(q_1 - 1)$

2) 2) $\mathcal{L}(q_1,q_2,\lambda) = q_1^2 + 6q_1q_2 + 2q_2^2 + 5q_1 + \lambda(2q_1 + 4q_2 - 46)$

$$\begin{cases} \frac{\partial U}{\partial q_1} = 2q_1 + 6q_2 + 5 + 2\lambda = 0 \\ \frac{\partial U}{\partial q_2} = 6q_1 + 4q_2 + 4\lambda = 0 \\ \frac{\partial U}{\partial \lambda} = 2q_1 + 4q_2 - 46 = 0 \end{cases} \Leftrightarrow \begin{cases} q_1 = 17 \\ q_2 = 3 \\ \lambda = -28{,}5 \end{cases}$$

$$H^* \begin{vmatrix} 0 & 2 & 4 \\ 2 & 2 & 6 \\ 4 & 6 & 4 \end{vmatrix} = 48 > 0 \quad \max$$

$U(17,3) = 698$

3) $\mathcal{L}(q_1,q_2;\lambda) = \quad + \lambda(p_1q_1 + 4q_2 - b)$

$$\begin{cases} \frac{\partial U}{\partial q_1} = 2q_1 + 6q_2 + 5 + p_1\lambda = 0 \\ \frac{\partial U}{\partial q_2} = 6q_1 + 4q_2 + 4\lambda = 0 \\ \frac{\partial U}{\partial \lambda} = p_1q_1 + 4q_2 - b = 0 \end{cases} \overset{(2,6)}{\Leftrightarrow} \begin{cases} \cancel{4}5 + p_1\lambda = 0 \\ 36 + 4\lambda = 0 \\ 2p_1 + 24 - b = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda = -9 \\ p_1 = \cancel{45/9}\ 5 \\ b = 34 \end{cases}$$

II $y_{k+2} + a y_{k+1} + 4y_k = 0$

1) $s^2 + as + 4 = 0 \qquad D = a^2 - 16$

$a^2 - 16 > 0 \Leftrightarrow a > 4 \vee a < -4$

$a \in]-\infty, -4[\cup]4, +\infty[$

2) a) $y_{k+2} + 4y_{k+1} + 4y_k = 0$

$s^2 + 4s + 4 = 0 \qquad D = 16 - 16 = 0$

$s_1 = \frac{-4}{2} = -2 \qquad s_2 = -2k$

$\Rightarrow y_g(k) = (-2)^k C_1 + (-2)^k k C_2$

$\lim_{k\to\infty} y_k = \pm\infty$ onstabiel

b) $\begin{cases} y_0 = 1 \Leftrightarrow \\ y_1 = 0 \Leftrightarrow \end{cases} \begin{cases} C_1 + \cancel{C_2} = 1 \\ -2C_1 - 2C_2 = 0 \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$

c) $y_p(k) = \sqrt{2}^k \left(A\cos\frac{\pi}{4}k + B\sin\frac{\pi}{4}k\right)$

d) $y_p(k) = k^2(-2)^k(Ak + B)$

1999-2000 : R1

II $y_{k+2} - 5y_{k+1} + 6y_k = -9k(2)^k$

1) $s^2 - 5s + 6 = 0 \qquad D = 25 - 24 = 1$

$$s_1 = \frac{5+1}{2} = 3 \qquad s_2 = \frac{5-1}{2} = 2$$

$$\Rightarrow y_g(k) = 3^k C_1 + 2^k C_2$$

2) $y_p(k) = (Ak + B)\, 2^k . k$

$\underline{k=-2}$: $-5 . 2(-A+B).(-1) + 6(-2A+B).(-2) = 8$

$\underline{k=0}$: $4(2A+B).2 + (-5).2(A+B) = 0$

$$\Leftrightarrow \begin{cases} -10A + 10B + 24A - 12B = 8 \\ 16A + 8B - 10A - 10B = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 14A - 2B = 8 \\ 6A - 2B = 0 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 3 \end{cases}$$

3) $y_k = 3^k C_1 + 2^k C_2 + \left(k^2 + 3k\right) 2^k$

1999-2000 : R2

II $y_{k+2} - 2y_{k+1} + y_k = 6k + 6$

1) $s^2 - 2s + 1 = 0 \qquad D = 4 - 4 = 0$

$$s_1 = \frac{2}{2} = 1 \qquad s_2 = k$$

$$\Rightarrow y_g(k) = 1^k C_1 + k C_2$$

2) $y_p(k) = (Ak + B)k^2$

$(A(k+2) + B)(k+2)^2 - 2\left[A(k+1) + B\right](k+1)^2 + (Ak+B)k^2 = 6k + 6$

$\Leftrightarrow (Ak + 2A + B)(k^2 + 4k + 4) - (2Ak + 2A + 2B)(k^2 + 2k + 1)$

$\quad + (Ak^3 + Bk^2) = 6k + 6$

$\Leftrightarrow \cancel{Ak^3} + \cancel{2Ak^2} + \cancel{Bk^2} + \cancel{4Ak^2} + 8Ak + \cancel{4Bk} + \cancel{4Ak} + 8A + 4B$

$\quad - \cancel{2Ak^3} - \cancel{2Ak^2} - \cancel{2Bk^2} - \cancel{4Ak^2} - \cancel{4Ak} - \cancel{4Bk} - 2Ak - 2A - 2B$

$\quad + \cancel{Ak^3} + \cancel{Bk^2} = 6k + 6$

$$\Leftrightarrow \begin{cases} 8A - 2A = 6 \\ 4B - 2A - 2B + 8A = 6 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 0 \end{cases}$$

3) $y_k = C_1 + kC_2 + k^3$

2de : 1999 - 2000

I $S(x,y) = \frac{32x}{2+3x} + \frac{16y}{4+y}$

1) $\omega(x,y) = \frac{32x}{2+3x} + \frac{16y}{4+y} - x - y$

2) $\mathcal{L}(x,y,\lambda) = \frac{32x}{2+3x} + \frac{16y}{4+y} - x - y + \lambda(x+y-10)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = \frac{(2+3x).32 - 32x.3}{(2+3x)^2} - 1 + \lambda = 0 & (1) \\ \frac{\partial \mathcal{L}}{\partial y} = \frac{16(4+y) - 16y}{(4+y)^2} - 1 + \lambda = 0 & (2) \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x + y - 10 = 0 & (3) \end{cases}$$

(1) = (2)

$32(2+\cancel{3x}-\cancel{3x})(4+y)^2 = 16(4+\cancel{y}-\cancel{y})(2+3x)^2$

$\Leftrightarrow$ ~~$64(16+8y+y^2) =$~~ $64(4+y)^2 = 64(2+3x)^2$

$\Leftrightarrow 4+y = 2+3x$

$\Leftrightarrow y = 3x-2$ (4)

(4) in (3) $x + 3x - 2 - 10 = 0$

$\Leftrightarrow 4x = 12$

$\Leftrightarrow x = 3$ (5)

(5) in (4)

$y = 9-2 = 7$

$\frac{\partial^2 \mathcal{L}}{\partial x^2} = \left(\frac{64}{(2+3x)^2}\right)' = \frac{-64.2(\cancel{2+3x}).3}{(2+3x)^{\cancel{4}\,3}} = \frac{-384}{(2+3x)^3} \Rightarrow \frac{-384}{1331}$

$\frac{\partial \mathcal{L}}{\partial y \partial x} = 0$

$\frac{\partial^2 \mathcal{L}}{\partial y^2} = \left(\frac{64}{(4+y)^2}\right)' = \frac{-64.2(\cancel{4+y})}{(4+y)^{\cancel{4}\,3}} = \frac{-128}{1331}$

$$H^* = \begin{vmatrix} 0 & 1 & 1 \\ 1 & \frac{-3884}{1331} & 0 \\ 1 & 0 & \frac{-128}{1331} \end{vmatrix} = (-1).\frac{-128}{1331} + \frac{3884}{1331} > 0 \quad \text{max}$$

$\omega(3,7) = 8909\,090$ BEF

II 1) $2s^2 - 4s + 4 = 0 \qquad D = 16 - 32 = -16$

$p = +1 \quad q = \cancel{2}\,1 \quad r = \sqrt{2} \quad \arctan \cancel{2}\,1 = \frac{\pi}{4}$

$y_g(k) = \sqrt{2}^{\,k}\left[C_1 \cos\frac{\pi}{4}k + C_2 \sin\frac{\pi}{4}k\right]$

2) $y_p(k) = \sqrt{2}^k (Ak + B)$

1^{ste} 2000-2001 · R1

I (i) 1) $s^2 - 2s + 1 = 0 \qquad D = 4 - 4 = 0$

$$s_1 = \frac{2}{2} = 1 \qquad s_2 = k$$

$$\Rightarrow y_g(k) = C_1 + kC_2$$

2) $y_p(k) = A\cos\frac{\pi}{2}k + B\sin\frac{\pi}{2}k$

$$\underbrace{A\cos\left(\frac{\pi}{2}(k+2)\right)} + B\underbrace{\sin\left(\frac{\pi}{2}(k+2)\right)} - 2A\underbrace{\cos\left(\frac{\pi}{2}(k+1)\right)} - 2B\underbrace{\sin\left(\frac{\pi}{2}(k+1)\right)}$$

$$+ A\cos\frac{\pi}{2}k + B\sin\frac{\pi}{2}k = \cos\frac{\pi}{2}k$$

$$\cos\left(k\frac{\pi}{2} + \pi\right) = \cos k\frac{\pi}{2} \cdot \cos\pi - \sin k\frac{\pi}{2} \cdot \sin\pi = -\cos k\frac{\pi}{2}$$

$$\sin\left(k\frac{\pi}{2} + \pi\right) = \sin k\frac{\pi}{2}\cos\pi + \sin\pi\cos k\frac{\pi}{2} = -\sin k\frac{\pi}{2}$$

$$\cos\left(k\frac{\pi}{2} + \frac{\pi}{2}\right) = \cos k\frac{\pi}{2}\cos\frac{\pi}{2} - \sin k\frac{\pi}{2}\sin\frac{\pi}{2} = -\sin k\frac{\pi}{2}$$

$$\sin\left(k\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin k\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos k\frac{\pi}{2} = +\cos k\frac{\pi}{2}$$

$$\Leftrightarrow -A\cos k\frac{\pi}{2} - B\sin k\frac{\pi}{2} + 2A\sin k\frac{\pi}{2} - 2B\cos k\frac{\pi}{2} + A\cos k\frac{\pi}{2}$$
$$+ B\sin k\frac{\pi}{2} = \cos k\frac{\pi}{2}$$

$$\Leftrightarrow \begin{cases} -A - 2B + A = 1 \\ -B + 2A + B = 0 \end{cases} \Leftrightarrow \begin{cases} B = -1/2 \\ A = 0 \end{cases}$$

3) $y_k = C_1 + kC_2 - \frac{1}{2}\sin\frac{\pi}{2}k$

(ii) $y_p(k) = (Ak + B)k^2$

II $D = 5200 \qquad q = z_2 + z_3$

$C_3 = 800$

$C_2 = 500 \qquad \frac{\tau_2}{z_2} = \frac{\tau_3}{z_3} = \frac{\tau_2 + \tau_3}{z_2 + z_3}$

$C_1 = 500$

1) $K(z_2, z_3) = \underset{1}{\text{bestelkosten}} + \underset{2}{\text{beheerskosten}} + \underset{3}{\text{voorraadtekort}}$

1) $C_1 \cdot \frac{D}{q} = C_1 \cdot \frac{D}{z_2 + z_3} = \frac{2\,600\,000}{z_2 + z_3}$

2) $\frac{1}{2} C_2 \cdot \frac{z_2^2}{z_2 + z_3} = \frac{250\, z_2^2}{z_2 + z_3}$

3) $\frac{1}{2} c_3 \frac{z_3^2}{z_2+z_3} = \frac{800 z_3^2}{z_2+z_3}$

$\Rightarrow K(z_2,z_3) = \frac{2\,600\,000 + 250 z_2^2 + 800 z_3^2}{z_2+z_3}$

2) $\begin{cases} \frac{\partial K}{\partial z_2} = \frac{-2600\,000}{(z_2+z_3)^2} + \frac{(z_2+z_3)500 z_2 - 250 z_2^2}{(z_2+z_3)^2} - \frac{-400 z_3^2}{(z_2+z_3)^2} = 0 \\ \frac{\partial K}{\partial z_3} = \frac{-2600\,000}{(z_2+z_3)^2} + \frac{-250 z_2^2}{(z_2+z_3)^2} + \frac{(z_2+z_3)800 z_3 - 400 z_3^2}{(z_2+z_3)^2} = 0 \end{cases}$

$\Leftrightarrow (z_2+z_3)500 z_2 - 2\cancel{5}0 z_2^2 - 4\cancel{0}0 z_3^2 = -2\cancel{5}0 z_2^2 + (z_2+z_3)800 z_3$

$\mp 40\cancel{0} z_3^2$

$\Leftrightarrow$

$(z_2+z_3)K(z_2;z_3) = 2600\,000 + 250 z_2^2 + 400 z_3^2$

$\begin{cases} \frac{\partial K}{\partial z_2} = 500 z_2 - K(z_2,z_3) = 0 & (1) \\ \frac{\partial K}{\partial z_3} = 800 z_3 - K(z_2,z_3) = 0 & (2) \end{cases}$

(1)-(2) $\Leftrightarrow 500 z_2 = 800 z_3$ (1)+(2) $500 z_2 + 800 z_3 - 2K(z_2,z_3) = 0$

$\Leftrightarrow z_2 = \frac{8}{5} z_3$ (3) $\Leftrightarrow$ *

~~(3) in (1) = $800 z_3$~~

* $(z_2+z_3)(500 z_2 + 800 z_3) - 500 z_2^2 - 800 z_3^2 = 5200\,000$

$\Leftrightarrow 800 z_2 z_3 + 500 z_2 z_3 = 5200\,000$

$\Leftrightarrow z_2 z_3 = 4000$ (4)

(3) (4) $\begin{cases} z_2 = 8/5\ z_3 \\ z_2 z_3 = 4000 \end{cases}$ $\frac{8}{5} z_3^2 = 4000$

$\Leftrightarrow \begin{cases} z_3 = 50 \vee (-50) \\ z_2 = 80 \end{cases}$

$q = z_2 + z_3 = 130$

$K(80,50) = 40\,000$

3) $\frac{5200}{130} = 40 \to$ # cycli

$= \frac{360}{40} = 9$ dagen

4) $\frac{\tau_3}{\tau_2+\tau_3} = \frac{z_3}{z_2+z_3} \Leftrightarrow \tau_3 = \frac{9 \cdot 50}{130} = 3,46 \simeq 3,5$ dagen

5) $\frac{\partial^2 K}{\partial z_2^2} = \frac{500}{z_2^* + z_3^*} = \frac{50}{13}$

1ste 2003-2004

II 1) $-2s^2 + 2as + \left(\frac{3}{2}a^2 + 2a + \frac{1}{2}\right) = 0$

$D = 4a^2 + 8(\ldots) = 4a^2 + 12a^2 + 16a + 4 = 0$

$= 16a^2 + 16a + 4$

$= 4(4a^2 + 4a + 1) = 4(2a+1)^2$

$s_1 = \frac{-2a + 2\sqrt{\quad}}{-4} = \ldots \quad 3a +$

$s_2 = \frac{-2a - 2\sqrt{\quad}}{-4} = 1 + \frac{1}{a}\sqrt{4a^2+4a+1}$

stabiel: $\left|1 - \frac{\sqrt{4a^2+4a+1}}{a}\right| < 1$

$\Leftrightarrow 0 < \frac{a - \sqrt{\quad}}{a} < 1$

$\Leftrightarrow 0 < a - \sqrt{\quad} < a$

$\Leftrightarrow -a < -\sqrt{\quad} < 0$

$\Leftrightarrow 0 < \sqrt{\quad} < a$

$\Leftrightarrow 0 < 4a^2 + 4a + 1 < a^2$

1ste 2003-2004

II 1) $-2s^2 + 2as + \left(\frac{3}{2}a^2 + 2a + \frac{1}{2}\right) = 0$

$D = 4a^2 + 8(\ldots) = 4a^2 + 12a^2 + 16a + 4$

$= 16a^2 + 16a + 4$

$= 4(4a^2 + 4a + 1)$

$= 4(2a+1)^2$

$s_1 = \frac{-2a + 2(2a+1)}{-4} = -\frac{a+1}{2}$

$s_2 = \frac{-2a - 2(2a+1)}{-4} = \frac{3a-1}{2}$

$\Rightarrow y_g(k) = C_1 s_1^k + C_2 s_2^k$

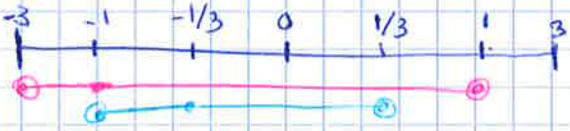
Stabiel: $\left|-\frac{a+1}{2}\right| < 1$ | $\left|\frac{3a-1}{2}\right| < 1$

$\Leftrightarrow -1 < \frac{-a+1}{2} < 1$ | $\Leftrightarrow -1 < \frac{3a-1}{2} < 1$

$\Leftrightarrow -2 < -a+1 < 2$ | $\Leftrightarrow -2 < 3a-1 < 2$

$\Leftrightarrow -1 < -a < 3$ | $\Leftrightarrow -3 < 3a < 1$

$\Leftrightarrow 1 > a > -3$ | $\Leftrightarrow -1 < a < 1/3$



$a \in \left]-1, 1/3\right[\rightarrow$ stabiel

D=0: $a = -1/2$

$s_1 = s_2 = -1/4 \rightarrow$ stabiel

2) $-2y_{k+2} + 10y_{k+1} + 48y_k$

$D = 100 + 384 = 484 \quad \sqrt{D} = 22$

$s_1 = \frac{-10+22}{-4} = -3$

$s_2 = \frac{-10-22}{-4} = 8$

$\Rightarrow y_g(k) = (-3)^k C_1 + 8^k C_2$

(a) $\cancel{(-3)^k (Ak+B)(D\cos\frac{\pi}{3}k + E\sin\frac{\pi}{3}k)\,k}$

$(-3)^k \left[(Ak+B)\cos\frac{\pi}{3}k + (Ck+D)\sin\frac{\pi}{3}k\right]$

(b) $(-3)^k (Ak+B)k + C$

2de 2008-2009

I i) $\begin{cases} \frac{\partial z}{\partial x} = (2x^2-4y^2)\cdot(-2x)\,e^{-(x^2+y^2)} + 4x\,e^{-(x^2+y^2)} = 0 & \cdot y \quad :2y \\ \frac{\partial z}{\partial y} = (2x^2-4y^2)\cdot(-2y)\,e^{-(x^2+y^2)} - 8y\,e^{-(x^2+y^2)} = 0 & \cdot x \quad \cdot x \end{cases}$

$\Leftrightarrow \begin{cases} 4xy - 8xy = 0 \\ 4xy + 2xy = 0 \end{cases}$ $\begin{cases} \frac{\partial z}{\partial x} = -4x\,e^{-(x^2+y^2)}(x^2-2y^2-1) = 0 \\ \frac{\partial z}{\partial y} = 4y\,e^{\cdots}(-x^2+2y^2-2) = 0 \end{cases}$

$\Leftrightarrow -x(x^2-2y^2-1) = y(-x^2+2y^2-2)$

$\Leftrightarrow -x^3 + 2xy^2 + x = -2y - xy^2$

$$\Leftrightarrow \begin{cases} x=0 \quad \vee \quad x^2-2y^2-1=0 \\ y=0 \quad \vee \quad -x^2+2y^2-2=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ -x^2+2y^2-2=0 \end{cases} \vee \begin{cases} y=0 \\ x^2-2y^2-1=0 \end{cases}$$

$$\vee \left.\begin{cases} x^2-2y^2-1=0 \\ -x^2+2y^2-2=0 \end{cases}\right\} \text{strijdig}$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=0 \\ y=1 \end{cases} \quad \begin{cases} x=0 \\ y=-1 \end{cases} \quad \begin{cases} y=0 \\ x=1 \end{cases} \quad \begin{cases} y=0 \\ x=-1 \end{cases}$$

2) $$\frac{\partial^2 z}{\partial x^2} = -4e^{\cdots}(x^2-2y^2-1) + (-4x)e^{\cdots}(\quad).(-2x) +$$

$$-4xe^{\cdots}(2x)$$

$$= -4e^{-x^2-y^2}(x^2-2y^2-1+8x^2-2x^2)$$

$$= -4e^{\cdots}(\cancel{7x^2 \; 2y^2})\;(-2x^4+4x^2+5x^2-2y^2-1)$$

$$\Rightarrow (0,-1): \frac{12}{e}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 \qquad \frac{\partial^2 z}{\partial y^2} = \frac{18}{e}$$

$$H^* = \begin{vmatrix} \frac{12}{e} & 0 \\ 0 & \frac{18}{e} \end{vmatrix} : \begin{matrix} H_1 > 0 \\ H_2 > 0 \end{matrix} \Rightarrow \text{min}$$

3) $$z\left(-\frac{1}{2}, y\right) = \left(\frac{1}{2} - 4y^2\right)e^{-(1/4+y^2)}$$

$$\frac{\partial z}{\partial y} = -8y\,e^{\cdots} - 2y\left(\frac{1}{2}-4y^2\right)e^{\cdots}$$

$$= e^{\cdots}(-8y - y + 8y^3)$$

$$= e^{-\frac{1}{4}-y^2}(8y^3-9y) = 0$$

$$\Leftrightarrow e\,y(8y^2-9) = 0$$

$$\Leftrightarrow y = 0 \quad \vee \quad 8y^2-9=0$$

$$\Leftrightarrow y = \frac{3}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\vee \quad y = -\frac{3\sqrt{2}}{4}$$

		$-\frac{3\sqrt{2}}{4}$		0		$\frac{3\sqrt{2}}{4}$	
y			$-$	0	$+$		
$8y^2-9$	$+$	0	$-$		$-$	0	$+$
	$-$	0	$+$	0	$-$	0	$+$
	↘		↗		↘		↗

c) niveaulijn: $z(x,y) = z(0,0) = 0$

$$2x^2 - 4y^2 = 0$$

$$\Leftrightarrow \quad x = \sqrt{2}\,y \quad \vee \quad x = -\sqrt{2}\,y$$

1ste 2009-2010: R1

I $f(x,y) = a^2x^2 + (6a-8)y^2 - 16x + 32y - 10$

1) (a)
$$\begin{cases} \dfrac{\partial f}{\partial x} = 2a^2x - 16 = 0 \\ \dfrac{\partial f}{\partial y} = 2(6a-8)y + 32 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \cancel{x = \pm\sqrt{8/a^2}} \\ \cancel{y = \dfrac{-8}{3a-4}} \end{cases} \qquad \Leftrightarrow \begin{cases} \cancel{x = 2a\sqrt{2}} \vee -2a\sqrt{2} \\ \cancel{y = \dfrac{-8}{3a-4}} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \dfrac{8}{a^2} \\ y = \dfrac{-8}{3a-4} \end{cases} \qquad P = \left(\dfrac{8}{a^2}, \dfrac{-8}{3a-4}\right)$$

(b) $H^* = \begin{vmatrix} 2a^2 & 0 \\ 0 & 2(6a-8) \end{vmatrix}$

H_1: $2a^2 > 0$

$H_2 = 2a^2 \cdot (12a - 16) = \cancel{\ldots}$

$a = 0 \quad \vee \quad a = \frac{4}{3}$

		0		4/3	
$2a^2$	+	0	+		
			−	0	+
	−	0	−	0	+

$a > 4/3 \rightarrow$ min

$a < 4/3 \rightarrow$ zadelpunt

2) $x^2 - 2y^2 - 16x + 32y - 10$

(b) $\mathcal{L}(x,y,\lambda) = x^2 - 2y^2 - 16x + 32y - 10 + \lambda(x - 3 - (y-8)^2)$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 2x - 16 + \lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = -4y + 32 - 2(y-8)\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = x - 3 - (y-8)^2 = 0 \end{cases}$$

$\Leftrightarrow -4(y-8) - 2(y-8)\lambda = 0$

$\Leftrightarrow -2(y-8)(2+\lambda) = 0$

$(2x-16)(-y+16) + 4y - 32 = 0$

$$\Leftrightarrow \begin{cases} y = 8 \\ x = 3 \\ \lambda = 10 \end{cases} \qquad \begin{cases} \lambda = -2 \\ x = 9 \\ 6 = (y-8)^2 \end{cases}$$

$(x, y) = (3, 8)$

$$H^*: \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -24 \end{vmatrix} \Rightarrow (-1) \cdot -24 = 24 > 0 \Rightarrow \max$$

3) a) $f(10, y) = 100 - 2y^2 - 160 + 32y - 10$

$$= -2y^2 + 32y - 70$$

$$= 2(-y^2 + 16y - 35)$$

$$f'_{x=10}(y) = -2y + 16 = 0$$

$$\Leftrightarrow y = 8 \Rightarrow \max (10, 8)$$

$$f''_{x=10}(y) = -2 < 0$$

(b) $\frac{\partial f}{\partial v}$ gezocht in $(10, 8)$ in richting (α, β) $\alpha < 0$

$$\frac{\partial f}{\partial v} = Df(x_0, y_0) \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= 4\alpha + 0\beta < 0$$

$\Rightarrow$ f beperkt tot het opgeg. gebied $\max (10, 8)$

Theorieopdrachten

Functies impliciet maken om vervolgens af te leiden

1 $x - y \neq 0 \iff y \neq x$

2 $\cancel{f(x,y) = (e^x - y)^{1/3}(x-y)^{-1/3} = z}$

$\cancel{G(x,y,z) = (e^x - y)^{1/3}(x-y)^{-1/3}}$

$\iff (e^x - y)(x-y)^{-1} = z^3$

$\iff (e^x - y) - z^3(x-y) = 0$

3 $G(x,y,z) = (e^x - y) - z^3(x-y)$

4 VW: $G(a,b,c) = 0$

$\frac{\partial G}{\partial y}(a,b) \neq 0$

$(0,1,0)$ VW1: ok

VW2: $\frac{\partial G}{\partial y} = -1 + z^3 = -1$ ok

$\left(4, 9, \sqrt[3]{\frac{e^4 - 9}{-5}}\right)$ VW1: $e^4 - 9 - \frac{e^4 - 9}{-5} \cdot -5 = 0$

VW2: $-1 + \frac{e^4 - 9}{-5} \neq 0$ ok

5 $dF: e^x dx - dy - z^3 dx + z^3 dy - 2z^2(x-y)dz$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\iff \left(e^x - z^3 - 2z^2(x-y)\frac{\partial z}{\partial x}\right)dx + \left[-1 + z^3 - 2z^2(x-y)\frac{\partial z}{\partial y}\right]dy = 0$$

$$\iff \begin{cases} e^x - z^3 - 2z^2(x-y)\frac{\partial z}{\partial x} = 0 \\ z^3 - 1 - 2z^2(x-y)\frac{\partial z}{\partial y} = 0 \end{cases}$$

$$\iff \begin{cases} \frac{\partial z}{\partial x} = \frac{z^3 - e^x}{-2z(x-y)} \\ \frac{\partial z}{\partial y} = \frac{1 - z^3}{-2z^2(x-y)} \end{cases}$$

Impliciet afleiden, niveaukromme & gradiëntvector

1 $dF = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$ | $g(x) = y$

$\Leftrightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} g'(x) dx = 0$ | $\frac{dy}{dx} = g'(x)$

$\Leftrightarrow dy = g'(x) dx$

$$\Leftrightarrow g'(x) = - \frac{\frac{\partial f}{\partial x}(x_0, f(x_0))}{\frac{\partial f}{\partial y}}$$

2 vector: $\left(1, -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}\right) \sim \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$

3 gradiëntvector $\perp$ richtingsvector

- rico's vermenigvuldigen $= -1$

gradiëntvector: $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \sim \left(1, \frac{\partial f / \partial y}{\partial f / \partial x}\right)$

$$\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} \cdot - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -1$$

- scalair product $= 0$

$$\frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} = 0$$

Differentie vgl

1 $s - a = 0$

$\Leftrightarrow s = a$

$\Rightarrow y_g(k) = a^k C_1$

$\rightarrow$ P.O. $C = 1$

2 $y_p(k) = c$

$c - ac = b$

$\Leftrightarrow c = \frac{b}{1-a}$

$y_k = a^k + \frac{b}{1-a}$

is AO.

3 $c(k+1) - c = b$

$\Leftrightarrow c = \frac{b}{k}$

Meetkundig bepalen van gebonden extrema

$\mathcal{L}(x_1, x_2, \lambda) = 2(x_1 - 1)^2 + 3x_2^2 - 1 + \lambda(x_1^2 + x_2^2 - 16)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 4(x_1 - 1) + 2x_1\lambda = 0 & \cdot x_2 \\ \frac{\partial \mathcal{L}}{\partial y} = 6x_2 + 2x_2\lambda = 0 & \cdot x_1 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x_1^2 + x_2^2 = 16 \end{cases}$$

$4x_1x_2 - 4x_2 - 6x_1x_2 = 0$

$\Leftrightarrow -2x_2(x_1 + 2) = 0$

$\Leftrightarrow x_2 = 0 \quad \vee \quad x_1 = -2$

$x_1 = 4 \vee x_1 = -4 \qquad x_2 = \sqrt{12} = 2\sqrt{3} \quad x_2 = -2\sqrt{3}$

$\Rightarrow (4,0) ; (-4,0) ; (-2, 2\sqrt{3}) ; (-2, -2\sqrt{3})$

Samengestelde Functie

* 1 $F : \mathbb{R}^3 \to \mathbb{R} : (t_1, t_2, t_3) \mapsto F(t_1, t_2, t_3)$

2 $\cancel{\frac{\partial F}{\partial x}}$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t_3}$$

* 1 $F : \mathbb{R}^2 \to \mathbb{R} : (x, y) \to F(x,y)$ met $F(x,y) = f(x, y, g(x,y))$

2 $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial x}{}$

Definities

Inleidende begrippen

* open bal

Een open bal met middelpunt $a(=(a_1, a_2, \ldots, a_n))$ en straal $\delta\,(>0)$ is de verzameling van alle punten x waarvoor de afstand a kleiner is dan δ

→ $B(a;\delta) = \{x \in \mathbb{R}^n \mid |x-a| < \delta\}$

* gesloten bal

Een gesloten bal met a en δ is de verzameling van x waarvoor de afstand tot a kleiner of gelijk aan δ

→ $\overline{B}(a;\delta) = \{x \in \mathbb{R}^n \mid |x-a| \leq \delta\}$

* gereduceerde bal

Een gereduceerde bal is een bal verminderd m/h middelpunt

- gered. open bal: $B^*(a;\delta) = \{x \in \mathbb{R}^n \mid 0 < |x-a| < \delta\}$
$= B(a;\delta) \setminus \{a\}$
- gered. gesloten bal: $\overline{B}^*(a;\delta) = \{x \in \mathbb{R}^n \mid 0 < |x-a| \leq \delta\}$
$= \overline{B}(a;\delta) \setminus \{a\}$

* inwendig punt

Een punt $a \in \mathbb{R}^n$ is een inwendig punt v/d verzameling A in $\mathbb{R}^n$ als er een $\varepsilon > 0$ bestaat zodanig dat de bal $B(a;\varepsilon)$ volledig gelegen is in A

* uitwendig punt

punt $a \in \mathbb{R}^n$ is uitwendig punt van A in $\mathbb{R}$ als $\varepsilon > 0$ bestaat zodanig $B(a;\varepsilon)$ volledig gelegen is in $\text{co}A = \mathbb{R}^n \setminus A$

* open verzameling

Een verzameling A is open als alle punten inwendige punten zijn

* gesloten verzameling

A heet gesloten als alle inwendige punten ook alle randpunten bevat

* begrensde verzameling

A heet begrensd als er een $r > 0$ bestaat zodat voor elk punt $a \in A$ geldt dat $|a| < r$

* compacte verzameling

A in $\mathbb{R}$ die zowel gesloten is als begrensd noemt men compact

* randpunt

Een punt $a \in \mathbb{R}^n$ is een randpunt v/d verzameling A in $\mathbb{R}^n$ als voor elke $\varepsilon > 0$ de bal $B(a, \varepsilon)$ zowel punten van A als punten die niet tot A behoren bevat

Functies van meerdere veranderlijken

* functie

Een relatie f v/e verzameling V naar een verzameling W is een functie als elk element x van V ten hoogste één bestemming heeft in W.

→ $f : V \to W : x \mapsto y = f(x)$

$f = \{(x, y) \mid y = f(x)\} \subseteq V \times W$

* definitieverzameling

De def. verzameling v/e functie f is de verzameling van alle herkomsten van f en is een deelverzameling van V:

→ $\text{def}(f) = \{x \in V \mid f(x) \in W\} \subseteq V$

* waardeverzameling of beeld

De verzameling v. alle bestemmingen van f is het beeld van f en is een deelverzameling van W

→ $\operatorname{im}(f) = \{y \in W \mid \exists x \in V \text{ waarvoor } y = f(x)\} \subseteq W$

* limieten

Is $f: \mathbb{R}^n \to \mathbb{R}$ gedefinieerd in een gered. open bal met middelpunt a dan

→ $\lim\limits_{x \to a} f(x) = L \ (\in \mathbb{R}) \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : x \in B^*(a;\delta) \Rightarrow f(x) \in B(L,\varepsilon)$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

met $x \to a$ voor $(x_1, x_2, \ldots, x_n) \to (a_1, \ldots, a_n)$

$$\begin{cases} x_1 \to a_1 \\ \vdots \\ x_n \to a_n \end{cases}$$

→ uniek: $\lim\limits_{x \to a} f(x) = L \wedge \lim\limits_{x \to a} f(x) = L' \Rightarrow L = L'$

* continu i/e punt a

Een functie $f: \mathbb{R}^n \to \mathbb{R}$ heet continu i/e punt $a \in \mathbb{R}^n$ a.s.a.

(i) $a \in \operatorname{def}(f)$

(ii) $\forall \varepsilon > 0, \exists \delta > 0;\ x \in B(a;\delta) \subseteq \operatorname{def}(f) \Rightarrow f(x) \in B(f(a);\varepsilon)$

* continu i/e open gebied

Een functie $f: \mathbb{R}^n \to \mathbb{R}$ heet continu i/e open gebied dat deelverzameling is van def(f) a.s.a. f continu is in elk punt van dit gebied

* continu i/e gesloten gebied

f heet continu i/e gesloten gebied ($\in$ def(f)) a.s.a f continu is i/h open gebied en voor een punt a o/d rand v/h gebied geldt:

(i) $a \in \operatorname{def}(f)$

(ii) $\forall \varepsilon > 0, \exists \delta > 0 : x \in B(a;\delta) \cap \operatorname{def}(f) \Rightarrow f(x) \in B(f(a);\varepsilon)$

* Stelling van Weierstraß

Is een functie $f: \mathbb{R}^n \to \mathbb{R}$ continu i/e compact gebied V van $\mathbb{R}^n$ dan is f begrensd en bereikt f tenminste eenmaal een grootste & een kleinste waarde in V

* ~~projectie~~ partieel afleidbaar

Een functie $f: \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto f(x,y)$ is partieel afleidbaar naar x i/h punt (a,b) als de partiële functie

$$f_{y=b} : \mathbb{R} \to \mathbb{R} : x \mapsto f_{y=b}(x) = f(x,b)$$

afleidbaar is in a.

* partiële afgeleide v/d eerste orde

De afgeleide $f'_{y=b}(a)$ noemt men de part. afgeleide v/d eerste orde van f naar x in het punt (a,b)

$$\to \frac{\partial f}{\partial x}(a,b) = \frac{d f_{y=b}}{dx}(a)$$

$$= \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \in \mathbb{R}$$

* Jacobmatrix

Als een functie $f: \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto f(x,y)$ partieel afleidbaar is naar x & y i/e punt (a,b) dan noemt men de rijmatrix de Jacobmatrix

$$\to Df(a,b) = \begin{pmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \end{pmatrix}$$

* Gradiënt (vector)

$$\to \nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b) \, ; \, \frac{\partial f}{\partial y}(a,b) \right)$$

* Stelling van Schwarz & Young

VW: $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$ continu in (x_0, y_0)

Stelling: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$

* Lemma

Vw: $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continue functies in (a,b)

Stelling: Er bestaan twee functies

$X:(h,k) \to X(h,k) \qquad Y:(h,k) \to Y(h,k)$

$\to f(a+h, b+k) = f(a,b) + h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b) + hX(h,k) + kY(h,k)$

met $\lim_{(h,k)\to(0,0)} X(h,k) = 0 = \lim_{(h,k)\to(0,0)} Y(h,k)$

* Kettingregel

Vw: (i) $f: \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto f(x,y)$

(ii) $x = g(t) \quad , \quad y = h(t)$

(iii) $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continu in (x,y)

(iv) g & h afleidbaar in t

Stelling: Dan geldt voor de samengestelde functie $F: \mathbb{R} \to \mathbb{R} : t \mapsto F(t)$

met $F(t) = f(x,y) = f(g(t), h(t))$

$\to F'(t) = \frac{\partial f}{\partial x}(x,y) \cdot g'(t) + \frac{\partial f}{\partial y}(x,y) \cdot h'(t)$

$\to \frac{dF}{dt}(t) = \frac{\partial f}{\partial x}(x,y) \frac{dg}{dt}(t) + \frac{\partial f}{\partial y}(x,y) \cdot \frac{dh}{dt}(t)$

Algemeen $F'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)\, g_i'(t)$

$\frac{dF}{dt_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}$

* Directionele afgeleide

Geg: $P_0(x_0, y_0) \qquad v = (\alpha, \beta)$

afg: $\begin{cases} x = x_0 + \alpha t \\ y = y_0 + \beta t \end{cases}$

$\to F(t) = f(x_0 + \alpha t, y_0 + \beta t)$

$F'(t) = \alpha \frac{\partial f}{\partial x}(x_0, y_0) + \beta \frac{\partial f}{\partial y}(x_0, y_0)$

$\to$ rico: $F'(0)$ voor $t = 0$

→ richtingsvector raaklijn

$$\left(\alpha, \beta, \alpha\frac{\partial f}{\partial x}(x_0,y_0) + \beta\frac{\partial f}{\partial y}(x_0,y_0)\right)$$

⇒ richtingsafgeleide

$\frac{\partial f}{\partial v}(x_0, y_0)$ met $v = (\alpha, \beta)$

→ Jacobmatrix: $\frac{\partial f}{\partial v}(x_0,y_0) = D f(x_0,y_0).v$ $\quad v = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

→ gradiëntvector: $\frac{\partial f}{\partial v}(x_0,y_0) = \nabla f(x_0,y_0).v$ $\quad v = (\alpha, \beta)$

→ eenheidsvector

$$\frac{\partial f}{\partial w}(x_0,y_0) = \frac{1}{|v|}\frac{\partial f}{\partial v}(x_0,y_0)$$

eig: Geg $f: \mathbb{R}^2 \to \mathbb{R}: (x,y) \mapsto f(x,y)$ die differentieerbaar is in een punt $(x_0,y_0) \in def(f)$

eig: 1) gradiëntvector in $(x_0,y_0) \perp$ raaklijn a/d niveaulijn in dat punt

2) gradiëntvector geeft richting aan waarin f het snelst toeneemt vanuit dat punt of m.a.w. is de richting v/d sterkste stijging

Bew 1) Geg - $\frac{\partial f}{\partial y}(x_0,y_0) \neq 0$

- $f(x,y) = k$ vgl. niveaulijn van f door (x_0,y_0) met niveau $k = f(x_0,y_0)$

→ rico raaklijn niveaukromme =

$$-\frac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}$$

→ rico rechte door (x_0,y_0) richting gradiëntvector

$$\nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)$$

$$\sim \left(1, \frac{\partial f/\partial y\,(x_0,y_0)}{\partial f/\partial x\,(x_0,y_0)}\right)$$

rico's: • $\dfrac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)} \cdot -\dfrac{\frac{\partial f}{\partial y}(x_0,y_0)}{\frac{\partial f}{\partial x}(x_0,y_0)} = -1 \rightarrow$ gestelde

• werken met richtingsvectoren

- richtingsvector raaklijn

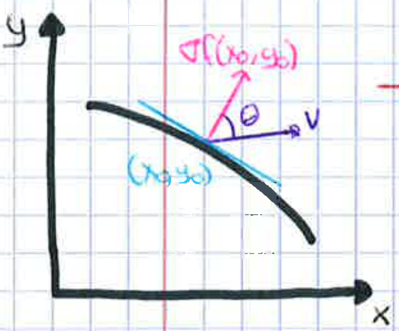
$$\left(1, -\dfrac{\frac{\partial f}{\partial x}(x_0,y_0)}{\frac{\partial f}{\partial y}(x_0,y_0)}\right) \sim \left(\frac{\partial f}{\partial y}(x_0,y_0), -\frac{\partial f}{\partial x}(x_0,y_0)\right)$$

- scalair product vector raaklijn & gradiëntvector

$$\frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial f}{\partial x}(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial f}{\partial y}(x_0,y_0) = 0$$

c) werken met eenheidsvector (lengte 1)

$v = (\alpha, \beta)$ $\quad |v|^2 = \alpha^2 + \beta^2 = 1$, zodat $\frac{\partial f}{\partial v}(x_0,y_0)$ max



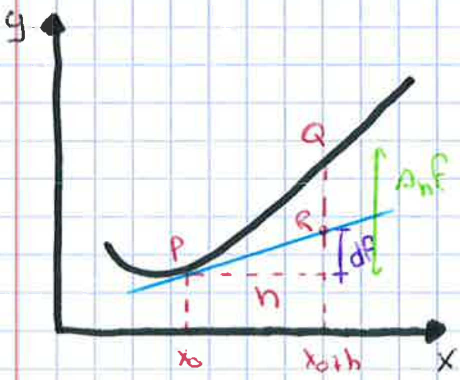
$$\rightarrow \frac{\partial f}{\partial v}(x_0,y_0) = \nabla f(x_0,y_0).v$$
$$= |\nabla f(x_0,y_0)|.\underbrace{|v|}_{=1}\cos\theta$$

$$\Rightarrow \begin{cases} \theta = 0 & \frac{\partial f}{\partial v}(x_0,y_0) = |\nabla f(x_0,y_0)| \rightarrow \max \\ \theta = \pi & \frac{\partial f}{\partial v}(x_0,y_0) = -|\nabla f(x_0,y_0)| \rightarrow \min \end{cases}$$

$\rightsquigarrow$ vector // gradiëntvector

$\begin{cases} \theta = 0 & \text{zelfde zin} \\ \theta = \pi & \text{tegengestelde zin} \end{cases}$

* Differentiaal v/e functie van $\mathbb{R}$ naar $\mathbb{R}$



• ~~dff~~ $df(x_0) : \mathbb{R} \rightarrow \mathbb{R} : h \mapsto f'(x_0)h$

• Taylor $\rightarrow \Delta_h f(x_0) \approx df(x_0)(h)$

$= f(x_0+h) - f(x_0) \approx f'(x_0).h$

• differentiaalfunctie f

vw: f afleidbaar

$df : \mathbb{R}^2 \rightarrow \mathbb{R} : (x,h) \mapsto df(x,h) = f'(x).h$

• samengestelde functie

$d(f\circ g)(t) = df(g(t)) = df(x) = f'(x)dx = f'(x)g'(t)dt$

* (continu) differentieerbaar i/e punt

VW $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continu in punt (a,b)

→ dan is f (continu) differentieerbaar in (a,b) en

- totale differentiaal in (a,b) voor toenames h,k van x,y

$$df((a,b),(h,k)) = h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)$$

- totale differentiaal(functie) in (a,b) is de (homogene) lineaire functie van twee veranderlijken

$$df_{(a,b)}: \mathbb{R}^2 \to \mathbb{R} : (h,k) \mapsto df_{(a,b)}(h,k) = h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)$$

* (continu) differentieerbaar in V (deelverzameling)

VW: $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continu in deelverzameling V van $\mathbb{R}^2$

→ $df: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : ((x,y),(h,k)) \mapsto df((x,y);(h,k)) =$

$$h\frac{\partial f}{\partial x}(x,y) + k\frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

→ $h = \Delta x = dx$, $k = \Delta y = dy$

* Totale differentiaal

Totale differentiaal van f is de som v/d partiële differentialen

→ $df((x,y);(\Delta x,\Delta y)) = df_{y=cte}(x,\Delta x) + df_{x=cte}(y,\Delta y)$

* differentiaal df als benadering v/d toename Δf

punt (a,b) en toenames h & k in x & y

$$\Delta f((a,b);(h,k)) = f(a+h, b+k) - f(a,b)$$

→ als h & k variëren:

$$\Delta f_{(a,b)}(h,k) = f(a+h,b+k) - f(a,b)$$

→ lemma: als $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continu

$$\Delta f_{(a,b)}(h,k) = df_{(a,b)}(h,k) + h\,X(h,k) + k\,Y(h,k)$$

$$= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + R(h,k)$$

met $\lim X(h,k) = 0 = \lim Y(h,k)$ h & k voldoende klein

$(R(h,k) = X(h,k) + Y(h,k))$

$\Rightarrow \Delta f_{(a,b)}(h,k) \approx df_{(a,b)}(h,k)$

! $\Delta f \neq df$, terwijl $\Delta x = dx$ en $\Delta y = dy$ als x, y onafh

* differentialen van constante functies

Een functie $f: \mathbb{R}^n \to \mathbb{R}: (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$ is een constante functie van n veranderlijken a.s.a. de totale differentiaalfunctie van f de constante functie op nul is

$\to f(x) = C,\ C \in \mathbb{R},\ \forall x \in V$

$\Leftrightarrow df(x,h) = 0 \quad \forall (x,h) \in V \times \mathbb{R}^n$

($V \subseteq \mathbb{R}^n$ gebied waar f differentieerbaar is)

* Impliciete functies $(n > 1)$

VW: $G: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}: (x,y) \mapsto G(x,y)$ differentieerbaar i/e open bal rond $(a,b) = (a_1, \ldots, a_n, b)$ met

- $G(a,b) = 0$
- $\frac{\partial G}{\partial y}(a,b) \neq 0$

Stelling: Dan bestaat er een functie $f: \mathbb{R}^n \to \mathbb{R}: x \mapsto f(x)$ in n veranderlijken, gedef. & differentieerbaar i/e open bal B rond a in $\mathbb{R}^n$ zodanig dat:

(i) $f(a) = b$

(ii) $G(x, f(x)) = 0 \quad \forall x \in B$

* berekenen impliciete afgeleide

Geg - $f: \mathbb{R} \to \mathbb{R}: x \mapsto f(x)$ impliciet gedef. door $G(x,y) = 0$

- $G: \mathbb{R}^2 \to \mathbb{R}$ differentieerbaar

$\to G(x, f(x)) = 0$

$\rightsquigarrow$ samengestelde functie $F: \mathbb{R} \to \mathbb{R}: x \mapsto G(x, f(x))$ is de constante nulfunctie

$0 = dF(x) = \frac{\partial G}{\partial x}(x, f(x))\,dx + \frac{\partial G}{\partial y}(x, f(x))\,f'(x)\,dx$

$$\Leftrightarrow \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} f'(x) = 0$$

$$\Leftrightarrow f'(x) = - \frac{\frac{\partial G}{\partial x}(x, f(x))}{\frac{\partial G}{\partial y}(x, f(x))}$$

$$\rightarrow \frac{dy}{dx}(x) = - \frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}$$

$\mathbb{R}^2 \rightarrow \mathbb{R}$: $\quad G(x,y,z) = 0$

$$G : \mathbb{R}^3 \rightarrow \mathbb{R} : (x,y,z) \mapsto \cancel{G(x,y,f(x,y))}\ G(x,y,z)$$

$$\rightarrow G(x,y,f(x,y)) = 0$$

$F : \mathbb{R}^2 \rightarrow \mathbb{R} : (x,y) \mapsto G(x,y,f(x,y))$ nulfunctie

$$0 = dF = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy + \frac{\partial G}{\partial z}\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)$$

$$= \left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial z}\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial G}{\partial y} + \frac{\partial G}{\partial z}\frac{\partial f}{\partial y}\right) dy$$

$$\Leftrightarrow \begin{cases} \frac{\partial G}{\partial x} + \frac{\partial G}{\partial z}\frac{\partial f}{\partial x} = 0 \\ \frac{\partial G}{\partial y} + \frac{\partial G}{\partial z}\frac{\partial f}{\partial y} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\partial f}{\partial x}(x,y) = - \dfrac{\frac{\partial G}{\partial x}(x,y,f(x,y))}{\frac{\partial G}{\partial z}(x,y,f(x,y))} \\ \\ \frac{\partial f}{\partial y}(x,y) = - \dfrac{\frac{\partial G}{\partial y}(x,y,f(x,y))}{\frac{\partial G}{\partial z}(x,y,f(x,y))} \end{cases}$$

* homogeen v/d graad

Een functie f, $f : \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n)$ heet homogeen v/d graad m, $m \in \mathbb{R}$ als

$$\rightarrow f(tx_1, \dots, tx_n) = t^m f(x_1, \dots, x_n) \quad \forall (x_1, \dots, x_n) \in \mathbb{R}^n, \forall t > 0$$

$\rightarrow$ eig: $f(x_1, x_2, \dots, x_n) = x_j^m f\left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \dots, 1, \dots, \frac{x_n}{x_j}\right)$

j-de plaats

Bew $f(tx_1, tx_2, \ldots, tx_n) = t^m f(x_1, x_2, \ldots, x_n)$

$\Leftrightarrow f(x_1, x_2, \ldots, x_n) = t^{-m} f(tx_1, tx_2, \ldots, tx_n)$

stel $t = x_j^{-1}$

$\Leftrightarrow f(x_1, \ldots, x_n) = x_j^m f\left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \ldots, 1, \ldots \frac{x_n}{x_j}\right)$

* eig. homogene functies

vw - f & partiële afgeleide tot en met de k^{de} orde continu

- f homogeen v/d graad m

eig: partiële afgeleide v/d eerste, tweede, ..., k^{de} orde homogeen v/d graad $m-1, m-2, \ldots, m-k$

* Stelling van Euler

vw: - $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ continu in x

- f homogeen v/d graad m

stelling $x_1 \frac{\partial f}{\partial x_1}(x) + \ldots + x_n \frac{\partial f}{\partial x_n}(x) = m f(x)$

→ matrixnotatie: $\mathbf{D}f(x) . x = m f(x)$

Optimalisatie van functies van meerdere veranderlijken

* kwadratische vorm

Een kwadratische vorm in $\mathbb{R}^n$ is een reëelwaardige functie v/d vorm $Q(x_1, \ldots, x_n) = \sum_{i=1} \sum_{i \leq j} a_{ij} x_i x_j$ waarin elke term een eenterm $\mathbb{R}^2$ naar $\mathbb{R}$ is v/d tweede graad

→ matrix:

$$(x_1 \; x_2 \; \cdots \; x_n) \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \cdots & \frac{1}{2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$= x^T A x$

→ A (unieke) sym. matrix

TB: 1) $A = \frac{1}{2}(B + B^T) = A^T$

2) $x^T A x = x^T B x$

Bew: 1) $A^T = \left(\frac{1}{2}(B + B^T)\right)^T$

$$= \frac{1}{2}\left((B)^T + (B^T)^T\right)$$

$$= \frac{1}{2}(B^T + B)$$

$$= A$$

2) $x^T A x = x^T \frac{1}{2}(B + B^T) x$

$$= \frac{1}{2}(x^T B x + \underbrace{x^T B^T x}_{1\times n \; n\times n \; n\times 1 = 1\times 1})$$

$$\Rightarrow x^T B^T x = (x^T B^T x)^T = x^T B x$$

$$= \frac{1}{2}(x^T B x + x^T B x)$$

$$= x^T B x$$

* Definiete kwadratische vormen

Geg: A een sym. $(n \times n)$-matrix

Def: - positief definiet $x^T A x > 0 \quad \forall x \neq 0$ in $\mathbb{R}^n$

- negatief " $x^T A x < 0$ "

- positief semidefiniet $x^T A x \geq 0$ "

- negatief " $x^T A x \leq 0$ "

- indefiniet $x^T A x > 0$ en < 0 voor waarden x in $\mathbb{R}^n$

* maximum

- (lokaal) max: $\exists r > 0 : \forall x \in B(a; r) \subseteq def(f) : f(x) \leq f(a)$

- globaal: $\forall x \in def(f) : f(x) \leq f(a)$

* minimum

- lokaal: $\exists r > 0 : \forall x \in B(a; r) \subseteq def(f) : f(x) \geq f(a)$

- globaal: $\forall x \in def(f) : f(x) \geq f(a)$

* nodige VW voor een extremum

Geg: - $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ gedef. in a

- a (lokaal) extremum

Stelling: $\begin{cases} \frac{\partial f}{\partial x_1}(a) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(a) = 0 \end{cases}$

Bew: $n = 2$ en a max

$\rightarrow \exists r > 0, \forall x \in B((a,b); r) \subseteq \mathrm{def}(f): f(x,y) \leq f(a,b)$

- y fixeren op b

$\exists r > 0, \forall x \in B((a,b); r) \subseteq \mathrm{def}(f): f(x,b) \leq f(a,b)$

- def. partiële functie

$\exists r > 0, \forall x \in]a-r, a+r[: f_{y=b}(x) \leq f_{y=b}(a)$

$\rightarrow$ a (lokaal) max voor functie $f_{y=b}$

- partiële afgeleide: $\frac{\partial f}{\partial x}$ bestaat in (a,b)

$\rightarrow f_{y=b}$ afleidbaar in a

$\Rightarrow f'_{y=b}(a) = 0 \Leftrightarrow \frac{\partial f}{\partial x}(a,b) = 0$

* stationair ∝ kritisch punt

Geg: $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ gedef. i/e punt a in def(f)

Def: a is een stationair of kritisch punt van f als

$\begin{cases} \frac{\partial f}{\partial x_1}(a) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(a) = 0 \end{cases}$

* Hessiaanmatrix

2x2 - matrix: $D^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ $\rightarrow$ hessiaan = $\det(D^2 f(x,y))$

* Verband Hessiaanmatrix & extremum

Geg - f & partiële afgeleide tot min. tweede orde continu
i/e open bal met middelpunt (a,b)
- (a,b) stationair punt

Stelling: 1) $D^2f(a,b)$ pos. def. lokaal min
2) " neg. def " max
3) " indefiniet zadelpunt

Bew: Taylor:

$$f(a+h,b+k) = f(a,b) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)\right)$$
$$+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\frac{\partial^2 f}{\partial x \partial y}(a,b) + k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right)$$
$$+ R(h,k)$$
$$= f(a,b) + \mathbf{D}f(a,b).h + \frac{1}{2}h^T\mathbf{D}^2f(a,b)h + R(h,k)$$

- def. stationair punt & verwaarlozen restterm

$$f(a+h,b+k) - f(a,b) \approx \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\frac{\partial^2 f}{\partial y \partial x} + k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right)$$

→ RL ≠ 0 ; teken LL bepaald door RL

$$\operatorname{sign}(\Delta f((a,b);(h,k))) = \operatorname{sign}(h^T\mathbf{D}^2f(a,b).h)$$

1) lokaal min: $\Delta f((a,b);(h,k)) > 0$
→ $D^2f(a,b)$ pos. def.

2) lokaal max: $\Delta(f((a,b);(h,k))) < 0$
→ neg. def

3) zadelpunt: $\Delta f((a,b);(h,k))$ wisselt van teken
→ $D^2f(a,b)$ indefiniet

* Betekenis Lagrangemultiplicator

$$\lambda^* = \frac{df^*(\alpha)}{d\alpha} = \lim_{\Delta\alpha \to 0} \frac{f^*(\alpha+\Delta\alpha) - f^*(\alpha)}{\Delta\alpha}$$

→ gevoeligheid v/d verandering v/d optimale functiewaarde

→ als α met één eenheid stijgt $\approx \lambda^*$ ↗ met één eenheid

Dynamische analyse: Differentievergelijkingen

* A.O. niet-homogen lineaire DV

  Geg: - $y_g(k)$ A.O. hom. DV

  - $y_p(k)$ P.O. niet-hom. DV

  eig: $y_k = y_g(k) + y_p(k)$

* lineair onafhankelijk

  Twee oplossingen v/e DV zijn lineair onafhankelijk, wanneer de ene oplossing niet kan geschreven worden als een veelvoud v/d andere

  $\rightarrow c_1 y_1(k) + c_2 y_2(k) = 0$

  $c_1 = c_2 = 0$